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BOLTZMANN SOLUTION OF COUPLED NONLINEAR EQUATIONS FOR MOISTURE CONTENT w(s) AND TEMPERATURE T(s), s=x/root(4t). BENCHMARK TEST I, CEN (2002).

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Report 2012:8

Göteborg 2012

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REPORT NO. 2012:8, ISSN 1652-9162
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Contents

Mathcad files for Benchmark Test I:

1. BENCHMARK TEST MODEL FOR COUPLED MOISTURE AND HEAT FLOW

There are many models for coupled moisture and heat flow in porous building materials. A question is how accurate the various models are. There is a need to compare the models with a solution with high and well documented accuracy. The model with its Mathcad files presented here tries to meet that need. It was originally presented in Dresden 2002 at a CEN meeting.

The solution is now used in a provisional European standard:

- European Provisional Standard prEN 15026, Hygrothermal Performance of Building Components and Building Elements – Assessment of Moisture Transfer by Numerical Simulation, 2005.

The presented model may be summarized in the following way:

- Highly non-linear coupled moisture and heat flow
- Highly unbalanced time scales between moisture content w and temperature T
- Any functional relations for flow coefficients and state functions are easily implemented
- Directly verified accuracy from with errors in the equations below 0.03 %. See page 34.
- Implemented in Mathcad. See pages 11-44.

2. EQUATIONS FOR COUPLED MOISTURE AND HEAT FLOW

2.1 Moisture and enthalpy balance equations

We consider one-dimensional coupled moisture and heat flow in a porous building material. The mass balance for moisture and heat read:

$$\frac{\partial w}{\partial t} = -\frac{\partial g}{\partial x}, \qquad x > 0, \qquad t > 0. \tag{2.1}$$

$$\frac{\partial h}{\partial t} = -\frac{\partial q}{\partial x}, \qquad x > 0, \qquad t > 0. \tag{2.2}$$

Here, w is the moisture content (kg/m^3), g the moisture flux (kg/(m^2,s)), h the enthalpy (J/m^3) and q the heat and enthalpy flux (J/(m^2,s)). These quantities are functions of the coordinate x and time t.

2.2 Liquid and vapor components

The moisture content w in the pores is divided into a *liq*uid and a *vap*or phase:

$$w = w_{\text{liq}} + w_{\text{vap}}, \qquad g = g_{\text{liq}} + g_{\text{vap}}. \tag{2.3}$$

The moisture flux g is also divided into a liquid and a vapor component.

2.3 Moisture and enthalpy flows

The liquid flux is by assumption driven by the suction (or negative pressure) P_{suc}, which is a given function of the water content w. The corresponding hydraulic conductivity $K(w)$ is also a given function of w. The vapor flow is driven by the water vapor pressure p in the gas phases with a moisture-dependent flow coefficient $\delta_p(w)$. We assume:

$$g_{\mathrm{liq}} = K(w) \cdot \frac{\partial P_{\mathrm{suc}}(w)}{\partial x}, \qquad g_{\mathrm{vap}} = -\delta_p(w) \cdot \frac{\partial p}{\partial x}. \tag{2.4}$$

The energy flux q involves heat conduction and convective enthalpy flux of liquid water and water vapor:

$$q = -\lambda(w) \cdot \frac{\partial T}{\partial x} + h_{\mathrm{liq}}(T) \cdot g_{\mathrm{liq}} + h_{\mathrm{vap}}(T) \cdot g_{\mathrm{vap}}. \tag{2.5}$$

The thermal conductivity depends on the water content w. The enthalpies are essentially proportional to the temperature T ($^{\circ}$C), (2.6).

4.4 Functional relations

We will need relations between our various quantities. For the enthalpies we have:

$$h_{\mathrm{solid}}(T) = c_{\mathrm{solid}} \cdot T, \qquad h_{\mathrm{liq}}(T) = c_p \cdot T, \qquad h_{\mathrm{vap}}(T) = h_{\mathrm{liq}}(T) + L_{\mathrm{ev}}(T). \tag{2.6}$$

Here, c_{solid} denotes the heat capacity of the solid porous material per *unit* volume (J/($^{\circ}$C,m^3)). The heat capacity of water is c_p=1000·4184 J/($^{\circ}$C,m^3), while $L_{\mathrm{ev}}(T)$ (J/m^3) is the latent heat of *ev*aporation of water.

The moisture content w depends on the relative humidity ϕ following the *sorp*tion isotherm of the considered porous material:

$$w = w_{\mathrm{sorp}}(\phi), \qquad \phi = \phi_{\mathrm{sorp}}(w), \qquad \phi = \frac{p}{p_{\mathrm{sat}}(T)}. \tag{2.7}$$

We will use the inverse to the sorption isotherm, i.e. the relative humidity ϕ as a function of w.

The enthalpy of the porous material is equal to the sum of enthalpies of the solid matrix, liquid water and water vapor:

$$h = h_{\mathrm{solid}}(T) + h_{\mathrm{liq}}(T) \cdot w_{\mathrm{liq}} + h_{\mathrm{vap}}(T) \cdot w_{\mathrm{vap}} = h_{\mathrm{solid}}(T) + h_{\mathrm{liq}}(T) \cdot w + L_{\mathrm{ev}}(T) \cdot w_{\mathrm{vap}}. \tag{2.8}$$

In the right-hand expression we have used (2.6), right.

We apply the general gas law to the water vapor in the gas part of the pore volume, V_{pore} - w/ρ_w (m^3/m^3).

$$\frac{R \cdot (273 + T) \cdot w_{\mathrm{vap}}}{M_w} = (V_{\mathrm{pore}} - w / \rho_w) \cdot p. \tag{2.9}$$

Here, R = 8.314 J/(mole,$^{\circ}$C)) is the gas constant, M_{w} = 0.018 kg/mole the molar mass of water, and ρ_w = 1000 kg/m^3 the density of liquid water.

We will use moisture content w and temperature T as basic state variables, so we need to express other variables as functions of w and T. From (2.7) we get for the water vapor pressure p:

$$p(w,T) = \phi_{\mathrm{sorp}}(w) \cdot p_{\mathrm{sat}}(T). \tag{2.10}$$

Combining (2.9) and (2.10), we get the water vapor content w_{vap} as function of w and T

$$w_{\text{vap}}(w,T)=\frac{M_w\cdot(V_{\text{pore}}-w/\rho_w)\cdot\phi_{\text{sorp}}(w)\cdot p_{\text{sat}}(T)}{R\cdot(273+T)}. \tag{2.11}$$

For the enthalpy we get from (2.8), right, and (2.11):

$$h(w,T)=h_{\text{solid}}(T)+h_{\text{liq}}(T)\cdot w+\frac{M_w\cdot(V_{\text{pore}}-w/\rho_w)\cdot\phi_{\text{sorp}}(w)}{R}\cdot\frac{p_{\text{sat}}(T)\cdot L_{\text{ev}}(T)}{273+T}. \tag{2.12}$$

The input to our model for heat and moisture involves the following functions for the considered porous material:

$$K(w),\qquad P_{\text{suc}}(w),\qquad \delta_p(w),\qquad \lambda(w),\qquad w=w_{\text{sorp}}(\phi)\quad\text{or}\quad \phi=\phi_{\text{sorp}}(w). \tag{2.13}$$

These functions are a free input in the model. For water we use the liquid and vapor enthalpy, which are related to the heat of evaporation:

$$h_{\text{liq}}(T),\qquad h_{\text{vap}}(T)=h_{\text{liq}}(T)+L_{\text{ev}}(T),\qquad p_{\text{sat}}(T). \tag{2.14}$$

We also need an expression for the saturation water vapor pressure.

2.5 Reformulation of the equations

The mass conservation equation (2.1) involves the moisture flux g. We have from (2.3), right, and (2.4) and (2.10):

$$g=K(w)\cdot\frac{\partial P_{\text{suc}}(w)}{\partial x}-\delta_p(w)\cdot\frac{\partial p}{\partial x}=K(w)\cdot\frac{dP_{\text{suc}}(w)}{dw}\cdot\frac{\partial w}{\partial x}-\delta_p(w)\cdot\frac{\partial}{\partial x}\left(\phi_{\text{sorp}}(w)\cdot p_{\text{sat}}(T)\right). \tag{2.15}$$

The moisture flux has the following form:

$$g=-K_w(w,T)\cdot\frac{\partial w}{\partial x}-K_T(w,T)\cdot\frac{\partial T}{\partial x}. \tag{2.16}$$

The coefficients before the gradients become:

$$\begin{aligned}K_w(w,T)&=-K(w)\cdot\frac{dP_{\text{suc}}}{dw}+\delta_p(w)\cdot\frac{d\phi_{\text{sorp}}}{dw}\cdot p_{\text{sat}}(T),\\ K_T(w,T)&=\delta_p(w)\cdot\phi_{\text{sorp}}(w)\cdot\frac{dp_{\text{sat}}}{dT}.\end{aligned} \tag{2.17}$$

The time derivative of the enthalpy in (2.2) may be expressed as derivatives of w and T:

$$\frac{\partial}{\partial t}\left[h(w,T)\right]=h_w(w,T)\cdot\frac{\partial w}{\partial t}+h_T(w,T)\cdot\frac{\partial T}{\partial t}=-\frac{\partial q}{\partial x},\qquad x>0,\qquad t>0. \tag{2.18}$$

Using (2.12), we get:

$$\begin{aligned}h_w(w,T)&=h_{\text{liq}}(T)+\frac{M_w}{R}\cdot\frac{d}{dw}\left[\left(V_{\text{pore}}-\frac{w}{\rho_w}\right)\cdot\phi_{\text{sorp}}(w)\right]\cdot\frac{p_{\text{sat}}(T)\cdot L_{\text{ev}}(T)}{273+T},\\ h_T(w,T)&=c_{\text{solid}}+c_w\cdot w+\frac{M_w\cdot(V_{\text{pore}}-w/\rho_w)\cdot\phi_{\text{sorp}}(w)}{R}\cdot\frac{d}{dT}\left[\frac{p_{\text{sat}}(T)\cdot L_{\text{ev}}(T)}{273+T}\right].\end{aligned} \tag{2.19}$$

The enthalpy conservation equation (2.18) involves the enthalpy flux q. Using (2.5), (2.4) and (2.10), this flux may be written:

$$q = -\lambda(w) \cdot \frac{\partial T}{\partial x} + h_{\text{liq}}(T) \cdot K(w) \cdot \frac{\partial P_{\text{suc}}(w)}{\partial x} - h_{\text{vap}}(T) \cdot \delta_p(w) \cdot \frac{\partial}{\partial x}\left(\phi_{\text{sorp}}(w) \cdot p_{\text{sat}}(T)\right). \quad (2.20)$$

The enthalpy flux has the following form:

$$q = -\lambda_w(w,T) \cdot \frac{\partial w}{\partial x} - \lambda_T(w,T) \cdot \frac{\partial T}{\partial x}. \quad (2.21)$$

The coefficients before the gradients become:

$$\begin{aligned} \lambda_w(w,T) &= -h_{\text{liq}}(T) \cdot K(w) \cdot \frac{dP_{\text{suc}}}{dw} + h_{\text{vap}}(T) \cdot \delta_p(w) \cdot \frac{d\phi_{\text{sorp}}}{dw} \cdot p_{\text{sat}}(T), \\ \lambda_T(w,T) &= \lambda(w) + h_{\text{vap}}(T) \cdot \delta_p(w) \cdot \phi_{\text{sorp}}(w) \cdot \frac{dp_{\text{sat}}}{dT}. \end{aligned} \quad (2.22)$$

2.6 Final equations

The final equations for the fluxes are now from (2.16) and (2.21):

$$g = -K_w(w,T) \cdot \frac{\partial w}{\partial x} - K_T(w,T) \cdot \frac{\partial T}{\partial x}, \quad (2.23)$$

$$q = -\lambda_w(w,T) \cdot \frac{\partial w}{\partial x} - \lambda_T(w,T) \cdot \frac{\partial T}{\partial x}, \qquad x \geq 0, \qquad t \geq 0. \quad (2.24)$$

The conservation equations for moisture and enthalpy become from from (2.1) and (2.18):

$$\frac{\partial w}{\partial t} = -\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left[K_w(w,T) \cdot \frac{\partial w}{\partial x} + K_T(w,T) \cdot \frac{\partial T}{\partial x}\right], \qquad x > 0, \qquad t > 0. \quad (2.25)$$

$$h_w(w,T) \cdot \frac{\partial w}{\partial t} + h_T(w,T) \cdot \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x} = \frac{\partial}{\partial x}\left[\lambda_w(w,T) \cdot \frac{\partial w}{\partial x} + \lambda_T(w,T) \cdot \frac{\partial T}{\partial x}\right]. \quad (2.26)$$

3. BOLTZMANN SOLUTION

Boltzmann solutions are applicable to certain diffusive partial differential equations in a semi-infinite region. Spatial derivatives of the second order and time derivatives of the first order occur in the diffusive partial differential equations. In particular cases we may get solutions for a single variable $s = x / \sqrt{4t}$. The partial differential equations in x and t become ordinary differential equations in the variable s. Only problems for very special boundary and initial conditions may be solved.

3.1 Boltzmann solution

We consider a step-response problem in a semi-infinite slab. The *in*itial moisture content and temperature at $t = 0$ are constant through the slab, and the *b*oundary moisture content and temperature at $x = 0$ are constant in time:

$$w(x,0) = w_{\text{in}}, \qquad T(x,0) = T_{\text{in}}, \qquad 0 < x < \infty,$$
$$w(0,t) = w_{\text{b}}, \qquad T(0,t) = T_{\text{b}}, \qquad 0 < t < \infty. \tag{3.1}$$

Here, w_{in}, T_{in}, w_{b} and T_{b} are any prescribed numbers.

We seek a solution of the following type:

$$w(x,t) = \tilde{w}(s), \qquad T(x,t) = \tilde{T}(s), \qquad s = \frac{x}{\sqrt{4\cdot t}}. \tag{3.2}$$

3.2 Equations for a Boltzmann solution

The solution of the type (3.2) shall satisfy the equations (2.23)-(2.26). We need the partial derivatives of the Boltzmann coordinate $s = s(x,t)$:

$$s = \frac{x}{\sqrt{4\cdot t}} \quad \Rightarrow \quad \frac{\partial s}{\partial x} = \frac{1}{\sqrt{4\cdot t}}, \qquad \frac{\partial s}{\partial t} = \frac{x}{\sqrt{4\cdot t}}\cdot\frac{-1}{2t} = \frac{-s}{2t}. \tag{3.3}$$

The moisture flux (2.23) becomes:

$$g(x,t) = -K_w(\tilde{w},\tilde{T})\cdot\frac{d\tilde{w}}{ds}\cdot\frac{1}{\sqrt{4t}} - K_T(\tilde{w},\tilde{T})\cdot\frac{d\tilde{T}}{ds}\cdot\frac{1}{\sqrt{4t}}. \tag{3.4}$$

The moisture flux becomes equal to a "Boltzmann moisture flux" $\tilde{g}(s)$ multiplied by a time factor:

$$g(x,t) = \tilde{g}(s)\cdot\frac{1}{\sqrt{4t}}, \qquad \tilde{g}(s) = -K_w(\tilde{w},\tilde{T})\cdot\frac{d\tilde{w}}{ds} - K_T(\tilde{w},\tilde{T})\cdot\frac{d\tilde{T}}{ds}. \tag{3.5}$$

The enthalpy flux (2.24) becomes:

$$q(x,t) = -\lambda_w(\tilde{w},\tilde{T})\cdot\frac{d\tilde{w}}{ds}\cdot\frac{1}{\sqrt{4t}} - \lambda_T(\tilde{w},\tilde{T})\cdot\frac{d\tilde{T}}{ds}\cdot\frac{1}{\sqrt{4t}}. \tag{3.6}$$

The enthalpy flux becomes equal to a "Boltzmann enthalpy flux" $\tilde{q}(s)$ multiplied by our time factor:

$$q(x,t) = \tilde{q}(s)\cdot\frac{1}{\sqrt{4t}}, \qquad \tilde{q}(s) = -\lambda_w(\tilde{w},\tilde{T})\cdot\frac{d\tilde{w}}{ds} - \lambda_T(\tilde{w},\tilde{T})\cdot\frac{d\tilde{T}}{ds}. \tag{3.7}$$

The moisture conservation equation (2.1) becomes with the use of (3.3) and (3.5), left:

$$\frac{\partial w}{\partial t} = -\frac{\partial g}{\partial x} \quad \Rightarrow \quad \frac{d\tilde{w}}{ds}\cdot\frac{-s}{2t} = -\frac{\partial}{\partial x}\left(\frac{\tilde{g}(s)}{\sqrt{4\cdot t}}\right) = -\frac{d\tilde{g}}{ds}\cdot\frac{1}{\sqrt{4\cdot t}}\cdot\frac{1}{\sqrt{4\cdot t}}. \tag{3.8}$$

The time factor cancels and we get:

$$2s\cdot\frac{d\tilde{w}}{ds} = \frac{d\tilde{q}}{ds}, \qquad 0 \le s < \infty. \tag{3.9}$$

The enthalpy conservation equation (2.18) becomes with the use of (3.3) and(3.7), left:

$$h_w(\tilde{w},\tilde{T})\cdot\frac{d\tilde{w}}{ds}\cdot\frac{-s}{2t} + h_T(\tilde{w},\tilde{T})\cdot\frac{d\tilde{T}}{ds}\cdot\frac{-s}{2t} = -\frac{d\tilde{q}}{ds}\cdot\frac{1}{\sqrt{4\cdot t}}\cdot\frac{1}{\sqrt{4\cdot t}}. \tag{3.10}$$

The time factor cancels and we get:

$$2s\cdot\left[h_w(\tilde{w},\tilde{T})\cdot\frac{d\tilde{w}}{ds}+h_T(\tilde{w},\tilde{T})\cdot\frac{d\tilde{T}}{ds}\right]=\frac{d\tilde{q}}{ds},\qquad 0\le s<\infty. \tag{3.11}$$

3.3 Four coupled ordinary differential equations

We have obtained four coupled first-order ordinary differential equations:

$$\tilde{g}(s)=-K_w(\tilde{w},\tilde{T})\cdot\frac{d\tilde{w}}{ds}-K_T(\tilde{w},\tilde{T})\cdot\frac{d\tilde{T}}{ds} \tag{3.12}$$

$$\tilde{q}(s)=-\lambda_w(\tilde{w},\tilde{T})\cdot\frac{d\tilde{w}}{ds}-\lambda_T(\tilde{w},\tilde{T})\cdot\frac{d\tilde{T}}{ds} \tag{3.13}$$

$$2s\cdot\frac{d\tilde{w}}{ds}=\frac{d\tilde{g}}{ds},\qquad 0\le s<\infty. \tag{3.14}$$

$$2s\cdot\left[h_w(\tilde{w},\tilde{T})\cdot\frac{d\tilde{w}}{ds}+h_T(w,T)\cdot\frac{d\tilde{T}}{ds}\right]=\frac{d\tilde{q}}{ds},\qquad 0\le s<\infty. \tag{3.15}$$

The first two equations (3.12)-(3.13) may be written as a matrix relation:

$$\begin{pmatrix}\tilde{g}(s)\\ \tilde{q}(s)\end{pmatrix}=-\begin{pmatrix}K_w & K_T\\ \lambda_w & \lambda_T\end{pmatrix}\cdot\frac{d}{ds}\begin{pmatrix}\tilde{w}(s)\\ \tilde{T}(s)\end{pmatrix}. \tag{3.16}$$

Using the inverse 2x2 matrix, we get:

$$\frac{d}{ds}\begin{pmatrix}\tilde{w}(s)\\ \tilde{T}(s)\end{pmatrix}=\frac{-1}{\mathrm{Det}_{K\lambda}}\cdot\begin{pmatrix}\lambda_T & -K_T\\ -\lambda_w & K_w\end{pmatrix}\cdot\begin{pmatrix}\tilde{g}(s)\\ \tilde{q}(s)\end{pmatrix},\qquad \mathrm{Det}_{K\lambda}=K_w\lambda_T-K_T\lambda_w. \tag{3.17}$$

The other two equations (3.14)-(3.15) may also be written as a matrix relation:

$$\frac{d}{ds}\begin{pmatrix}\tilde{g}(s)\\ \tilde{q}(s)\end{pmatrix}=2s\cdot\begin{pmatrix}1 & 0\\ h_w & h_T\end{pmatrix}\cdot\frac{d}{ds}\begin{pmatrix}\tilde{w}(s)\\ \tilde{T}(s)\end{pmatrix}=\frac{-2s}{\mathrm{Det}_{K\lambda}}\cdot\begin{pmatrix}1 & 0\\ h_w & h_T\end{pmatrix}\cdot\begin{pmatrix}\lambda_T & -K_T\\ -\lambda_w & K_w\end{pmatrix}\cdot\begin{pmatrix}\tilde{g}(s)\\ \tilde{q}(s)\end{pmatrix}. \tag{3.18}$$

In the right-hand matrix expression, (3.17) is inserted.

From (3.17)-(3.18) we have obtained the following four coupled ordinary differential equations:

$$\begin{cases}\dfrac{d\tilde{w}}{ds}=-\dfrac{\lambda_T}{\mathrm{Det}_{K\lambda}}\cdot\tilde{g}(s)+\dfrac{K_T}{\mathrm{Det}_{K\lambda}}\cdot\tilde{q}(s)\\[2ex] \dfrac{d\tilde{T}}{ds}=\dfrac{\lambda_w}{\mathrm{Det}_{K\lambda}}\cdot\tilde{g}(s)-\dfrac{K_w}{\mathrm{Det}_{K\lambda}}\cdot\tilde{q}(s)\\[2ex] \dfrac{d\tilde{g}}{ds}=-2s\cdot\dfrac{\lambda_T}{\mathrm{Det}_{K\lambda}}\cdot\tilde{g}(s)+2s\cdot\dfrac{K_T}{\mathrm{Det}_{K\lambda}}\cdot\tilde{q}(s)\\[2ex] \dfrac{d\tilde{q}}{ds}=-2s\cdot\dfrac{h_w\lambda_T-h_T\lambda_w}{\mathrm{Det}_{K\lambda}}\cdot\tilde{g}(s)+2s\cdot\dfrac{h_wK_T-h_TK_w}{\mathrm{Det}_{K\lambda}}\cdot\tilde{q}(s)\end{cases}\qquad 0\le s<\infty. \tag{3.19}$$

Here, we use the following functions, which are defined in(2.17), (2.22), (2.19) and (3.17), right:

$$K_w = K_w(\tilde{w},\tilde{T}), \qquad K_T = K_T(\tilde{w},\tilde{T}), \qquad \lambda_w = \lambda_w(\tilde{w},\tilde{T}),$$
$$\lambda_T = \lambda_T(\tilde{w},\tilde{T}), \qquad h_w = h_w(\tilde{w},\tilde{T}), \qquad h_T = h_T(\tilde{w},\tilde{T}), \tag{3.20}$$
$$\mathrm{Det}_{K\lambda} = \mathrm{Det}_{K\lambda}(\tilde{w},\tilde{T}) = K_w(\tilde{w},\tilde{T}) \cdot \lambda_T(\tilde{w},\tilde{T}) - K_T(\tilde{w},\tilde{T}) \cdot \lambda_w(\tilde{w},\tilde{T}).$$

The equation system (3.19) is of the following type:

$$\frac{d}{ds}\begin{pmatrix} \tilde{w}(s) \\ \tilde{T}(s) \\ \tilde{g}(s) \\ \tilde{q}(s) \end{pmatrix} = \begin{pmatrix} F_w(s,\tilde{w},\tilde{T},\tilde{g},\tilde{q}) \\ F_T(s,\tilde{w},\tilde{T},\tilde{g},\tilde{q}) \\ F_g(s,\tilde{w},\tilde{T},\tilde{g},\tilde{q}) \\ F_q(s,\tilde{w},\tilde{T},\tilde{g},\tilde{q}) \end{pmatrix}, \qquad 0 \le s < \infty. \tag{3.21}$$

Here, the right-hand expressions $F_{\mathrm{sub}}(s,\tilde{w},\tilde{T},\tilde{g},\tilde{q})$, sub = w, T, g and q, are given functions of s and the four state functions. This kind of coupled ordinary differential equations is easily solved with high accuracy in mathematical computer programs such as Mathcad. We need four boundary conditions to have a solution.

3.4 Boundary conditions for Boltzmann solution

The initial conditions (3.1), upper line, and the boundary condition (3.1), lower line, correspond to infinite s and to $s = 0$:

$$t \to 0, \; x > 0 \quad \Rightarrow \quad s \to \infty; \qquad x = 0, \; t > 0 \quad \Rightarrow \quad s = 0:$$
$$\tilde{w}(\infty) = w_{\mathrm{in}}, \qquad \tilde{T}(\infty) = T_{\mathrm{in}}, \qquad \tilde{w}(0) = w_{\mathrm{b}}, \qquad \tilde{T}(0) = T_{\mathrm{b}}. \tag{3.22}$$

The moisture content and the temperature at $s = 0$ are prescribed. We have to choose the moisture and heat flux at $s = 0$ so that the prescribed initial values are obtained:

$$\tilde{g}(0) \text{ and } \tilde{q}(0) \text{ to be chosen so that } \tilde{w}(\infty) = w_{\mathrm{in}} \text{ and } \tilde{T}(\infty) = T_{\mathrm{in}}. \tag{3.23}$$

Boltzmann solution of coupled nonlinear equations for moisture content w(s) and temperature T(s), s=x/root(4t). Benchmark test I, CEN. Dresden 2002.

J. Claesson

Input data.

Initial and boundary conditions

$\phi in := 0.5 \quad Tin := 20 \qquad \phi b := 0.95 \quad Tb := 30$

Constants

$Tref := 20 \qquad TKref := 273.15 + Tref \qquad \rho w := 1000 \qquad R := 8.314 \qquad Mw := 0.018$

Sorption isotherm.

$$w\phi(\phi) := \frac{146}{\left[1 + \left(-8\cdot 10^{-8}\cdot\frac{R\cdot TKref\cdot\rho w}{Mw}\cdot\ln(\phi)\right)^{1.6}\right]^{0.375}}$$

$$\phi w(w) := e^{\frac{-Mw}{R\cdot TKref\cdot\rho w}\cdot 0.125\cdot 10^{8}\cdot\left[\left(\frac{146}{w}\right)^{\frac{1}{0.375}}-1\right]^{0.625}}$$

Initial and boundary moisture content:

$win := w\phi(\phi in) \quad wb := w\phi(\phi b) \qquad win = 42.922 \qquad wb = 129.021$

Water retention curve.

$$wPsuc(Psuc) := \frac{146}{\left[1 + \left(8\cdot 10^{-8}\cdot Psuc\right)^{1.6}\right]^{0.375}} \qquad Psucw(w) := 0.125\cdot 10^{8}\cdot\left[\left(\frac{146}{w}\right)^{\frac{1}{0.375}} - 1\right]^{0.625}$$

Vapor diffusion coefficient.

$$\delta p(w) := \frac{Mw}{R\cdot TKref}\cdot\frac{26.1\cdot 10^{-6}}{200}\cdot\frac{1-\frac{w}{146}}{0.503\cdot\left(1-\frac{w}{146}\right)^{2} + 0.497}$$

Liquid water permeability.

$$K(w) := e^{-39.2619+0.0704\cdot(w-73)-1.742\cdot 10^{-4}\cdot(w-73)^{2}-2.7953\cdot 10^{-6}\cdot(w-73)^{3}-1.1566\cdot 10^{-7}\cdot(w-73)^{4}+2.5969\cdot 10^{-9}\cdot(w-73)^{5}}$$

Formula for p.sat(T).

$$Epsat(T) := \frac{19.625\cdot T}{270.1 + T} - \frac{0.00463\cdot T^{2}}{142.15 + T} \qquad psat(T) := 610.8\cdot e^{Epsat(T)}$$

Heat of evaporation.

$$Lev(T) := 10^{6}\cdot\left[2.5016 - \frac{T}{422} - \left(\frac{T}{342}\right)^{4}\right]$$

Heat content (enthalpy) of water and water vapor.

$hl(T) := 4184\cdot T \qquad hv(T) := hl(T) + Lev(T)$

Thermal conductivity.

$$\lambda(w) := 1.5 + 0.0158\cdot w$$

Water vapor content

$Vp := 0.146$ (pore volume)

$$wvap(w,T) := \left(Vp - \frac{w}{\rho w}\right)\cdot\phi w(w)\cdot\frac{Mw\cdot psat(T)}{R\cdot(273.15 + T)}$$

Total energy (enthalpy)

$$h(w,T) := 1.824\cdot 10^{6}\cdot T + hl(T)\cdot w + Lev(T)\cdot wvap(w,T)$$ (only used for control)

Capacity functions (derivatives of h(w,T)).

$$hw(w,T) := hl(T) + Lev(T)\cdot\frac{Mw\cdot psat(T)}{R\cdot(273.15 + T)}\cdot\left[-\frac{\phi w(w)}{\rho w} + \left(Vp - \frac{w}{\rho w}\right)\cdot\left(\frac{d}{dw}\phi w(w)\right)\right]$$

$$hT(w,T) := 1.824\cdot 10^{6} + 4184\cdot w + \left(Vp - \frac{w}{\rho w}\right)\cdot\frac{\phi w(w)\cdot Mw}{R}\cdot\frac{d}{dT}\left(\frac{Lev(T)\cdot psat(T)}{273.15 + T}\right)$$

Auxilliary functions.

Functions for flow coefficients:

$$Kw(w,T) := K(w)\cdot\frac{d}{dw}(-Psucw(w)) + psat(T)\cdot\delta p(w)\cdot\frac{d}{dw}(\phi w(w))$$

$$\lambda w(w,T) := -hl(T)\cdot K(w)\cdot\frac{d}{dw}(Psucw(w)) + hv(T)\cdot psat(T)\cdot\delta p(w)\cdot\frac{d}{dw}(\phi w(w))$$

$$KT(w,T) := \delta p(w)\cdot\phi w(w)\cdot\frac{d}{dT}psat(T) \qquad \lambda T(w,T) := \lambda(w) + hv(T)\cdot\delta p(w)\cdot\phi w(w)\cdot\frac{d}{dT}psat(T)$$

Determinant and R-functions:

$$DetK\lambda(w,T) := Kw(w,T)\cdot\lambda T(w,T) - KT(w,T)\cdot\lambda w(w,T)$$

$$R_1(w,T) := \frac{\lambda T(w,T)}{DetK\lambda(w,T)} \qquad R_2(w,T) := \frac{-KT(w,T)}{DetK\lambda(w,T)} \qquad R_3(w,T) := \frac{-\lambda w(w,T)}{DetK\lambda(w,T)}$$

$$R_4(w,T) := \frac{Kw(w,T)}{DetK\lambda(w,T)} \qquad R_5(w,T) := \frac{\lambda T(w,T)}{DetK\lambda(w,T)} \qquad R_6(w,T) := \frac{-KT(w,T)}{DetK\lambda(w,T)}$$

$$R_7(w,T) := \frac{hw(w,T)\cdot\lambda T(w,T) - hT(w,T)\cdot\lambda w(w,T)}{DetK\lambda(w,T)} \qquad R_8(w,T) := \frac{-hw(w,T)\cdot KT(w,T) + hT(w,T)\cdot Kw(w,T)}{DetK\lambda(w,T)}$$

Calculations.

Estimates of moisture and temperature penetration depths.

$$Lpenw := \sqrt{Kw(wb,Tb)} \qquad LpenT := \sqrt{\frac{\lambda T(wb,Tb)}{hT(wb,Tb)}} \qquad Lpenw = 8.469\times 10^{-6} \qquad LpenT = 1.224\times 10^{-3}$$

Coupled differential equations for w(s), T(s), g(s) and q(s).

$$D(s,y) := (-1)\cdot\begin{bmatrix} R_1(y_0,y_1)\cdot y_2 + R_2(y_0,y_1)\cdot y_3 \\ R_3(y_0,y_1)\cdot y_2 + R_4(y_0,y_1)\cdot y_3 \\ 2\cdot s\cdot\left(R_5(y_0,y_1)\cdot y_2 + R_6(y_0,y_1)\cdot y_3\right) \\ 2\cdot s\cdot\left(R_7(y_0,y_1)\cdot y_2 + R_8(y_0,y_1)\cdot y_3\right) \end{bmatrix}$$

(y.0=w)
(y.1=T)
(y.2=g)
(y.3=q)

Estimate of fluxes at s=0.

$$gbinit := Kw(wb,Tb)\cdot\frac{wb - win}{Lpenw} + KT(wb,Tb)\cdot\frac{Tb - Tin}{LpenT} \qquad qbinit := \lambda w(wb,Tb)\cdot\frac{wb - win}{Lpenw} + \lambda T(wb,Tb)\cdot\frac{Tb - Tin}{LpenT}$$

$gbinit = 7.296\times 10^{-4}$ $qbinit = 2.907\times 10^{4}$

Solution in moisture penetration region.

Range: $smax := 3\cdot Lpenw$ $smax = 2.541\times 10^{-5}$

Values at s=0: $y_0 := wb$ $y_1 := Tb$

Iterativ choice of fluxes g(0) and q(0).

$y_2 := 0.8131\cdot gbinit$ $y_3 := 0.815\cdot qbinit$

$$y = \begin{pmatrix} 129.021 \\ 30 \\ 5.932\times 10^{-4} \\ 2.369\times 10^{4} \end{pmatrix}$$

Runge-Kutta solver: $N := 100$ $Z := rkfixed(y, 0, smax, N, D)$

$i := 0..N$ $s_i := Z_{i,0}$ $w_i := Z_{i,1}$ $T_i := Z_{i,2}$ $g_i := Z_{i,3}$ $q_i := Z_{i,4}$

Iterations for choice of g(0) to get w(smax)=win: $w_N = 42.919$ $win = 42.922$

Continued solution for thermal penetration region.

Constant mosture content: $winf := w_N$ $winf = 42.919$

Range: $smax1 := 3\cdot LpenT$ $smax1 = 3.671\times 10^{-3}$ Initial values: $y1_0 := T_N$ $y1_1 := q_N$

Differential equations for T(s) and q(s) for constant moisture content:

$$D1(s1,y1) := (-1)\cdot\begin{pmatrix} \dfrac{1}{\lambda T(winf, y1_0)}\cdot y1_1 \\ 2\cdot s1\cdot\dfrac{hT(winf, y1_0)}{\lambda T(winf, y1_0)}\cdot y1_1 \end{pmatrix}$$

(y1.0=T)
(y1.1=q)

$$y1 = \begin{pmatrix} 29.743 \\ 2.36\times 10^{4} \end{pmatrix}$$

Runge-Kutta solver: $N1 := 100$ $Z1 := rkfixed(y1, smax, smax1, N1, D1)$

$i := 0..N1$ $s1_i := Z1_{i,0}$ $T1_i := Z1_{i,1}$ $q1_i := Z1_{i,2}$

Iterations for choice of q(0) to get T(smax1)=Tin: $T1_{N1} = 20.001$ $Tin = 20$

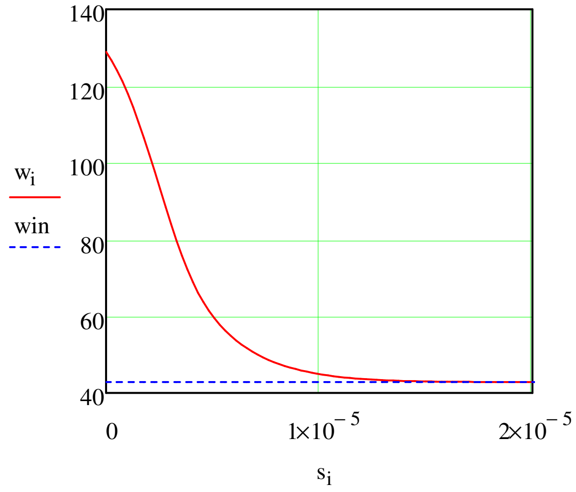
Diagrams for the result.

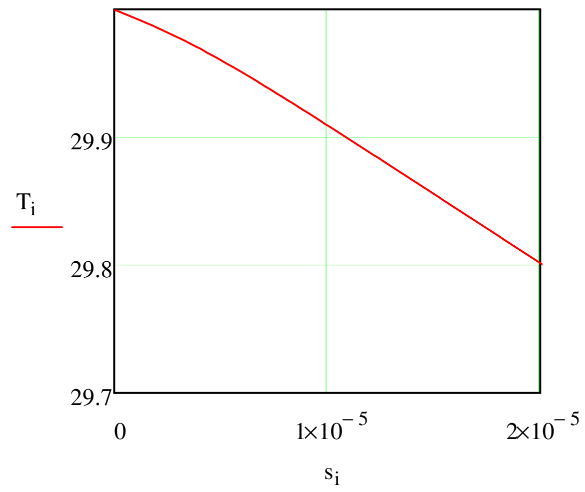
$i := N + 1 .. N + N1 \qquad s_i := s1_{i-N} \qquad w_i := winf \qquad T_i := T1_{i-N} \qquad g_i := 0 \qquad q_i := q1_{i-N}$

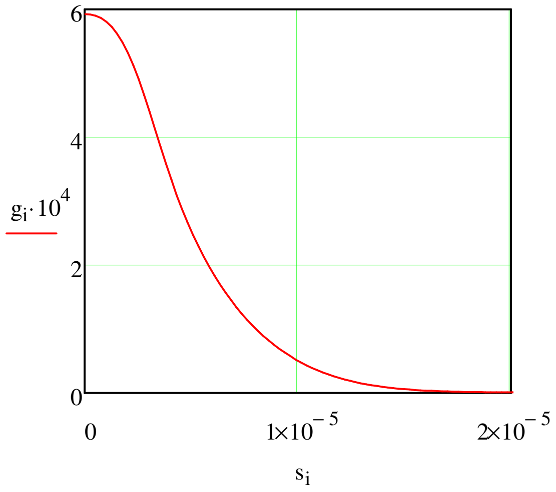
$i := 0 .. N \qquad i1 := 0 .. N + N1$

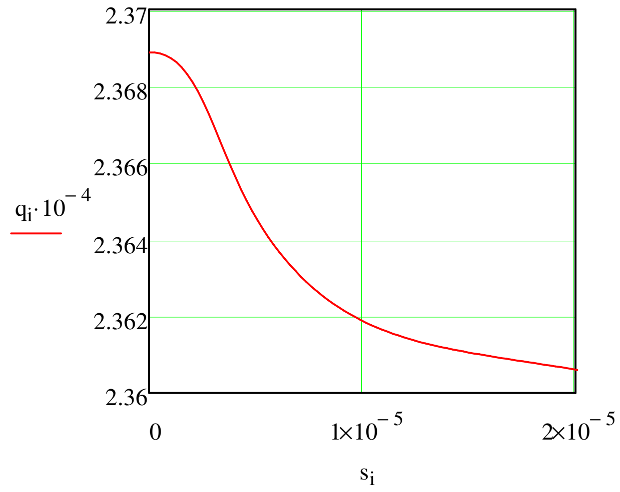
Benchmark test I, CEN.

Moisture penetration region

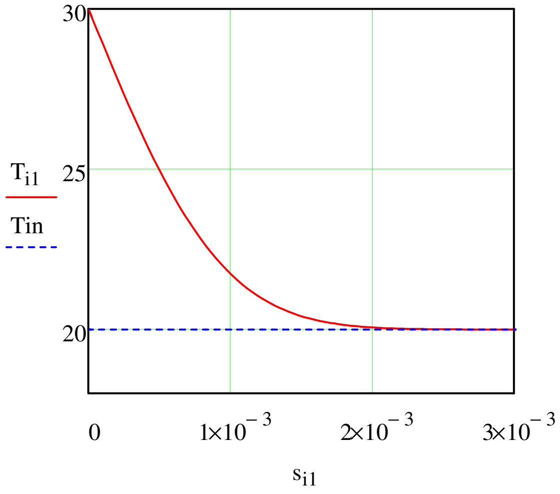


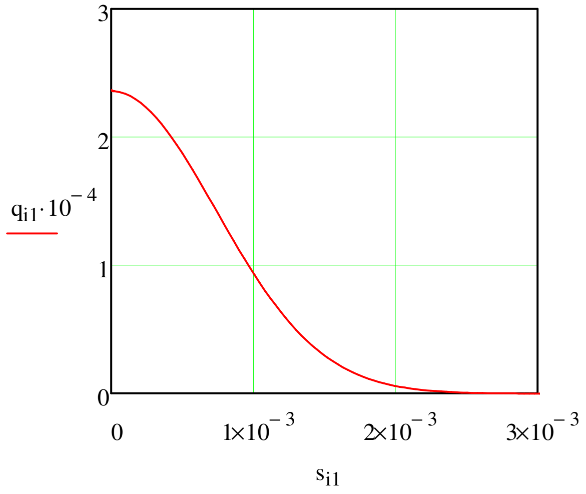






Thermal penetration region





Functions of x for a few t.

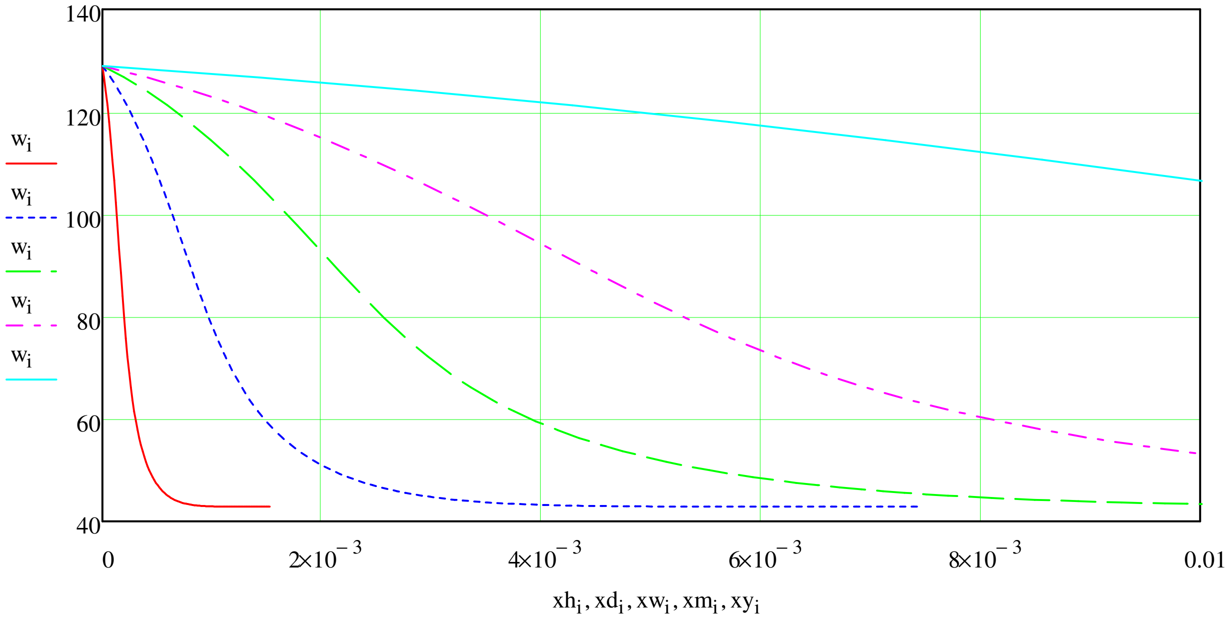
$xh := \sqrt{3600 \cdot s}$ (hour)

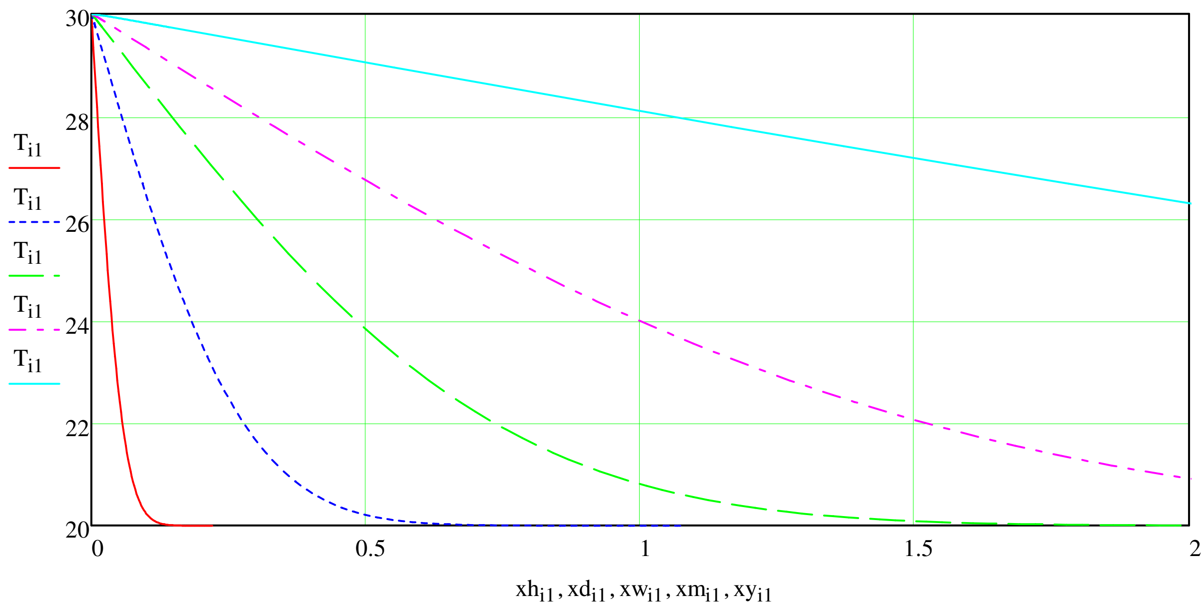
$xd := \sqrt{24 \cdot 3600 \cdot s}$ (day)

$xw := \sqrt{7 \cdot 24 \cdot 3600 \cdot s}$ (week)

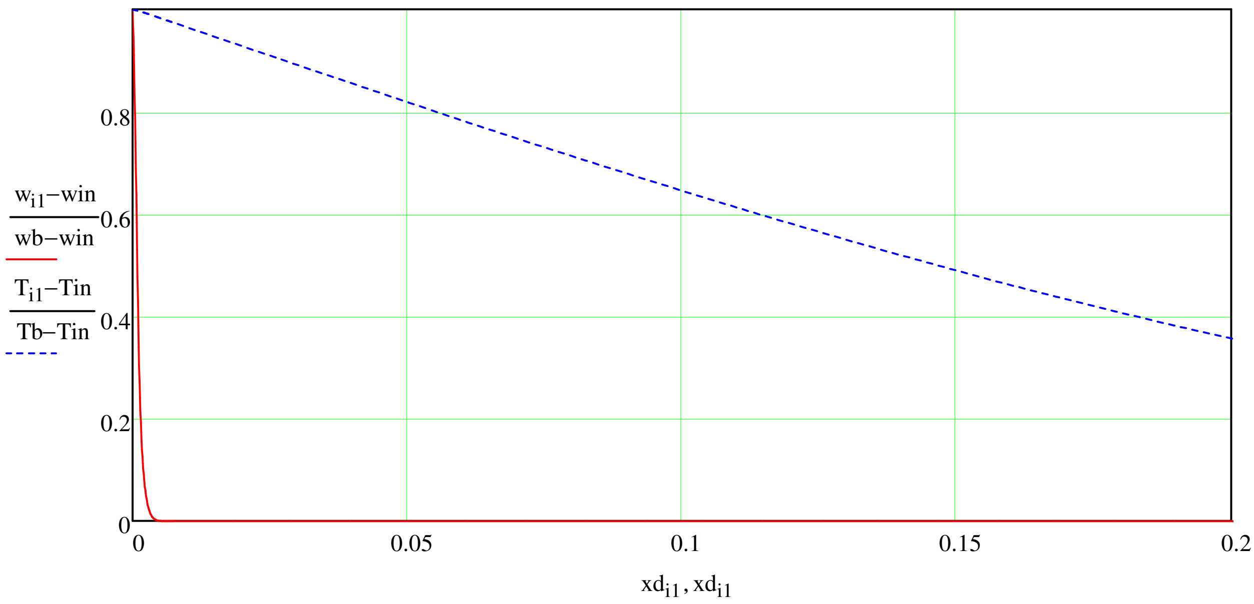
$xm := \sqrt{30 \cdot 24 \cdot 3600 \cdot s}$ (month)

$xy := \sqrt{365 \cdot 24 \cdot 3600 \cdot s}$ (year)

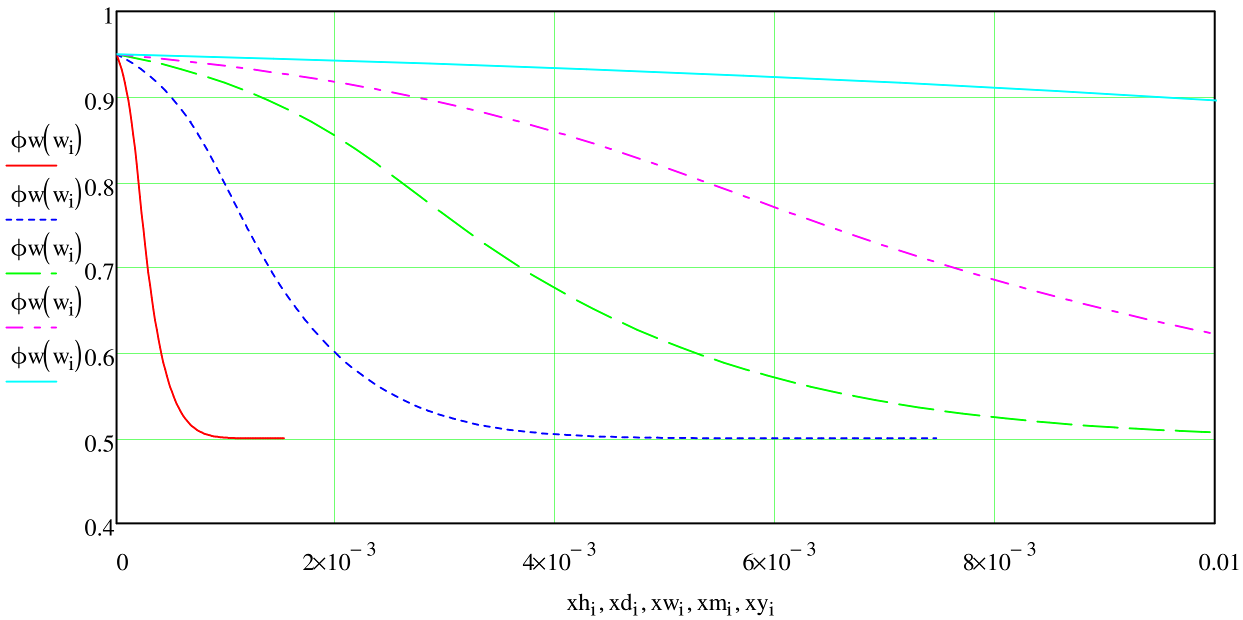




Moisture and temperature in the same diagram (t=24 h).



Relative humidity.



$$wf(sf) := \begin{cases} \text{linterp}(s, w, sf) \\ w_{200} \text{ if } sf \ge s_{200} \end{cases} \qquad Tf(sf) := \begin{cases} \text{linterp}(s, T, sf) \\ T_{200} \text{ if } sf \ge s_{200} \end{cases} \qquad s_{200} = 3.671 \times 10^{-3}$$

$$I := 10 \qquad i := 0..I \qquad x_i := \frac{i}{I} \cdot 0.1$$

$$wh_i := wf\left(\frac{x_i}{\sqrt{4 \cdot 3600}}\right) \qquad wd_i := wf\left(\frac{x_i}{\sqrt{4 \cdot 3600 \cdot 24}}\right) \qquad ww_i := wf\left(\frac{x_i}{\sqrt{4 \cdot 3600 \cdot 24 \cdot 7}}\right)$$

$$wm_i := wf\left(\frac{x_i}{\sqrt{4 \cdot 3600 \cdot 24 \cdot 30}}\right) \qquad wy_i := wf\left(\frac{x_i}{\sqrt{4 \cdot 3600 \cdot 24 \cdot 365}}\right)$$

x =

	0
0	0
1	0.01
2	0.02
3	0.03
4	0.04
5	0.05
6	0.06
7	0.07
8	0.08
9	0.09
10	0.1

wh =

	0
0	129.021
1	42.919
2	42.919
3	42.919
4	42.919
5	42.919
6	42.919
7	42.919
8	42.919
9	42.919
10	42.919

wd =

	0
0	129.021
1	42.965
2	42.919
3	42.919
4	42.919
5	42.919
6	42.919
7	42.919
8	42.919
9	42.919
10	42.919

ww =

	0
0	129.021
1	52.391
2	43.439
3	42.929
4	42.919
5	42.919
6	42.919
7	42.919
8	42.919
9	42.919
10	42.919

wm =

	0
0	129.021
1	83.109
2	53.288
3	45.732
4	43.569
5	43.035
6	42.934
7	42.921
8	42.919
9	42.919
10	42.919

wy =

	0
0	129.021
1	119.668
2	106.561
3	90.833
4	75.747
5	64.901
6	57.823
7	53.197
8	50.025
9	47.832
10	46.281

$$Matw := \text{augment}(x, wh, wd, ww, wm, wy)$$

$I := 10 \qquad i := 0..I \qquad x_i := \frac{i}{I} \cdot 5$

$$Th_i := Tf\left(\frac{x_i}{\sqrt{4 \cdot 3600}}\right) \qquad Td_i := Tf\left(\frac{x_i}{\sqrt{4 \cdot 3600 \cdot 24}}\right) \qquad Tw_i := Tf\left(\frac{x_i}{\sqrt{4 \cdot 3600 \cdot 24 \cdot 7}}\right)$$

$$Tm_i := Tf\left(\frac{x_i}{\sqrt{4 \cdot 3600 \cdot 24 \cdot 30}}\right) \qquad Ty_i := Tf\left(\frac{x_i}{\sqrt{4 \cdot 3600 \cdot 24 \cdot 365}}\right)$$

x =

	0
0	0
1	0.5
2	1
3	1.5
4	2
5	2.5
6	3
7	3.5
8	4
9	4.5
10	5

Th =

	0
0	30
1	20.001
2	20.001
3	20.001
4	20.001
5	20.001
6	20.001
7	20.001
8	20.001
9	20.001
10	20.001

Td =

	0
0	30
1	22.494
2	20.212
3	20.007
4	20.001
5	20.001
6	20.001
7	20.001
8	20.001
9	20.001
10	20.001

Tw =

	0
0	30
1	26.641
2	23.84
3	21.914
4	20.815
5	20.295
6	20.09
7	20.024
8	20.006
9	20.002
10	20.001

Tm =

	0
0	30
1	28.348
2	26.749
3	25.286
4	24.005
5	22.931
6	22.069
7	21.408
8	20.924
9	20.583
10	20.354

Ty =

	0
0	30
1	29.536
2	29.056
3	28.579
4	28.107
5	27.642
6	27.186
7	26.739
8	26.303
9	25.881
10	25.472

$MatT := augment(x, Th, Td, Tw, Tm, Ty)$

Matw =

	0	1	2	3	4	5
0	0	129.021	129.021	129.021	129.021	129.021
1	0.01	42.919	42.965	52.391	83.109	119.668
2	0.02	42.919	42.919	43.439	53.288	106.561
3	0.03	42.919	42.919	42.929	45.732	90.833
4	0.04	42.919	42.919	42.919	43.569	75.747
5	0.05	42.919	42.919	42.919	43.035	64.901
6	0.06	42.919	42.919	42.919	42.934	57.823
7	0.07	42.919	42.919	42.919	42.921	53.197
8	0.08	42.919	42.919	42.919	42.919	50.025
9	0.09	42.919	42.919	42.919	42.919	47.832
10	0.1	42.919	42.919	42.919	42.919	46.281

MatT =

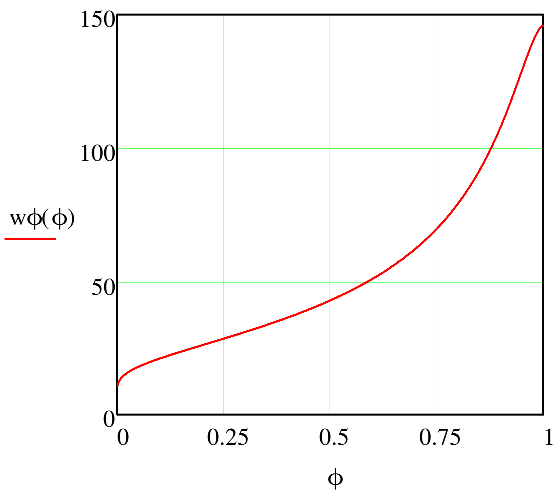
	0	1	2	3	4	5
0	0	30	30	30	30	30
1	0.5	20.001	22.494	26.641	28.348	29.536
2	1	20.001	20.212	23.84	26.749	29.056
3	1.5	20.001	20.007	21.914	25.286	28.579
4	2	20.001	20.001	20.815	24.005	28.107
5	2.5	20.001	20.001	20.295	22.931	27.642
6	3	20.001	20.001	20.09	22.069	27.186
7	3.5	20.001	20.001	20.024	21.408	26.739
8	4	20.001	20.001	20.006	20.924	26.303
9	4.5	20.001	20.001	20.002	20.583	25.881
10	5	20.001	20.001	20.001	20.354	25.472

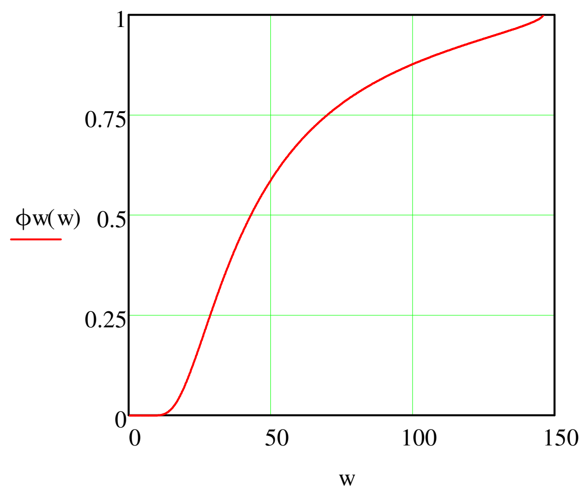
$Tref := 20 \qquad TKref := 273.15 + Tref \qquad \rho w := 1000 \qquad R := 8.314 \qquad Mw := 0.018$

Sorption isotherm.

$$w\phi(\phi) := \frac{146}{\left[1 + \left(-8\cdot 10^{-8}\cdot\frac{R\cdot TKref\cdot\rho w}{Mw}\cdot\ln(\phi)\right)^{1.6}\right]^{0.375}}$$

$$\phi w(w) := e^{\frac{-Mw}{R\cdot TKref\cdot\rho w}\cdot 0.125\cdot 10^{8}\cdot\left[\left(\frac{146}{w}\right)^{\frac{1}{0.375}}-1\right]^{0.625}}$$





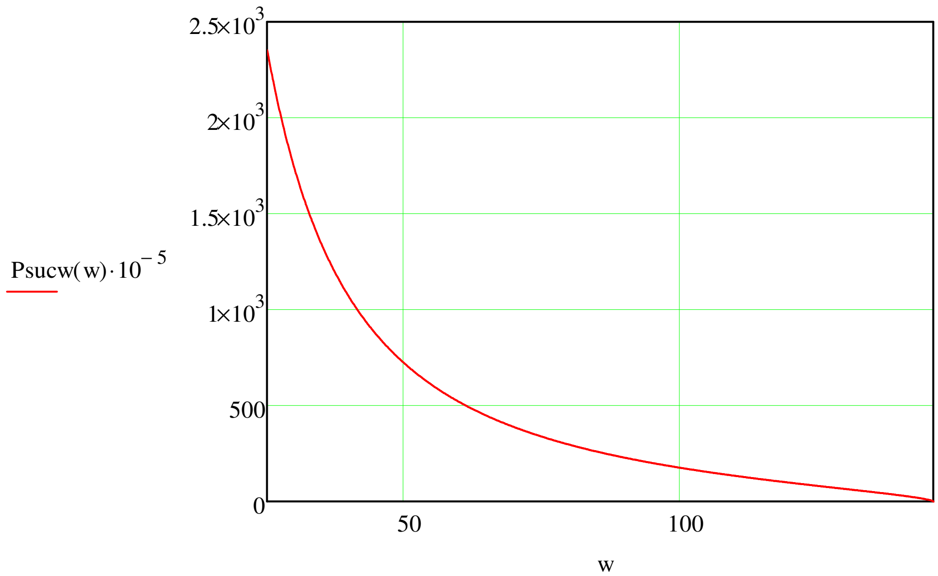
$w\phi\left(10^{-10}\right) = 5.323$ (!!) $\qquad w\phi(0.01) = 13.972 \qquad w\phi(0.5) = 42.922 \qquad \phi w(42.922) = 0.5$

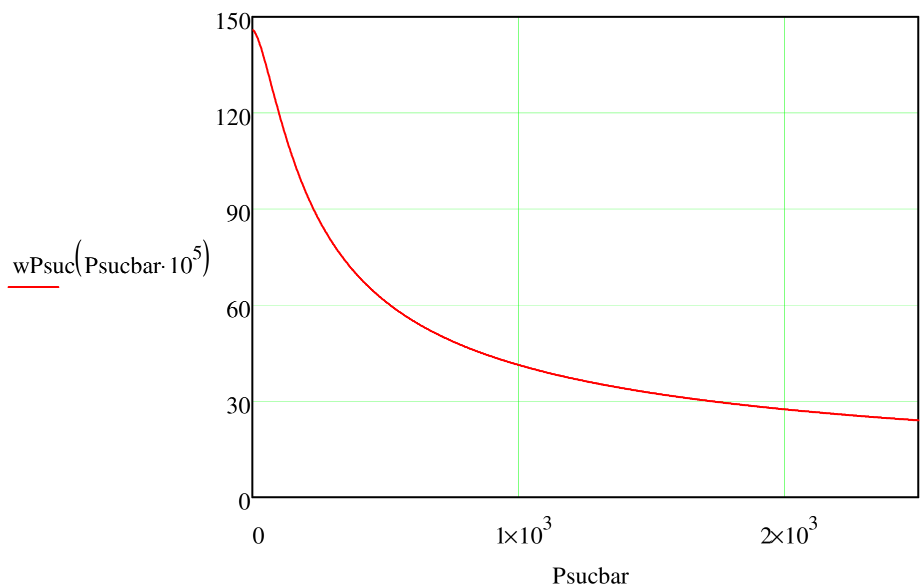
$w\phi(0.95) = 129.021$

Water retention curve.

$$wPsuc(Psuc) := \frac{146}{\left[1 + \left(8\cdot 10^{-8}\cdot Psuc\right)^{1.6}\right]^{0.375}}$$

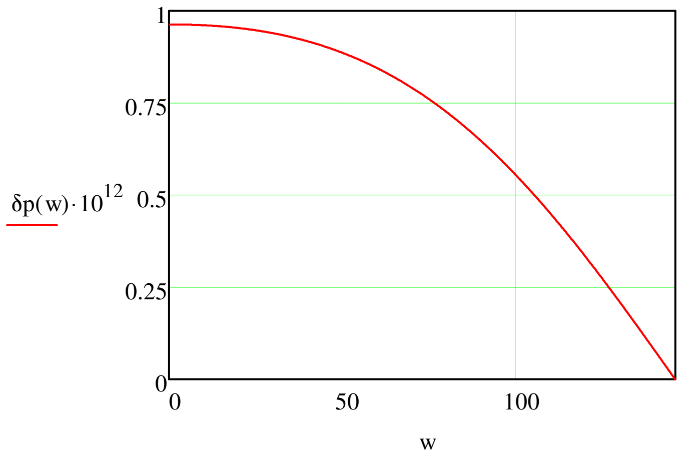
$$Psucw(w) := 0.125\cdot 10^{8}\cdot\left[\left(\frac{146}{w}\right)^{\frac{1}{0.375}} - 1\right]^{0.625}$$

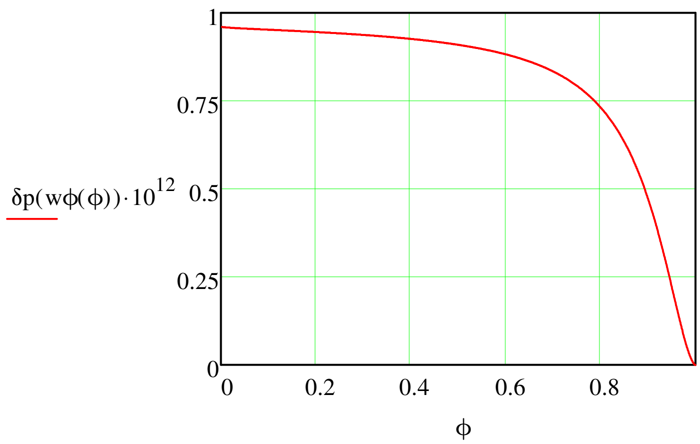




Vapor diffusion.

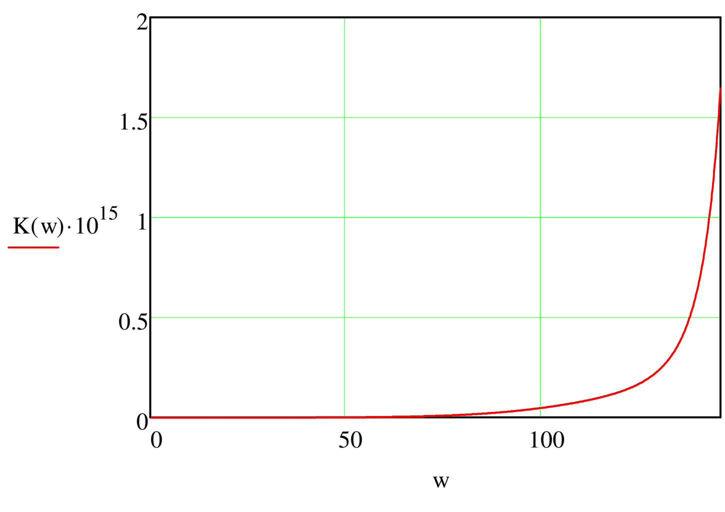
$$\delta p(w) := \frac{Mw}{R \cdot TKref} \cdot \frac{26.1 \cdot 10^{-6}}{200} \cdot \frac{1 - \frac{w}{146}}{0.503 \cdot \left(1 - \frac{w}{146}\right)^2 + 0.497}$$

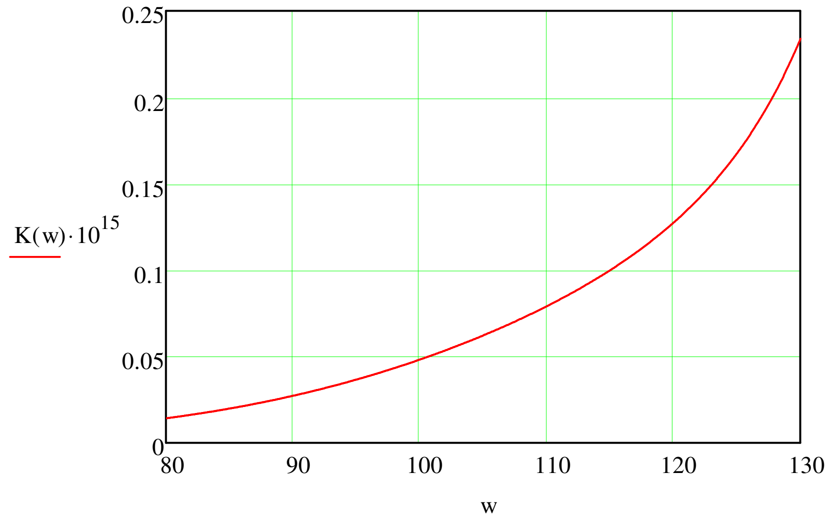


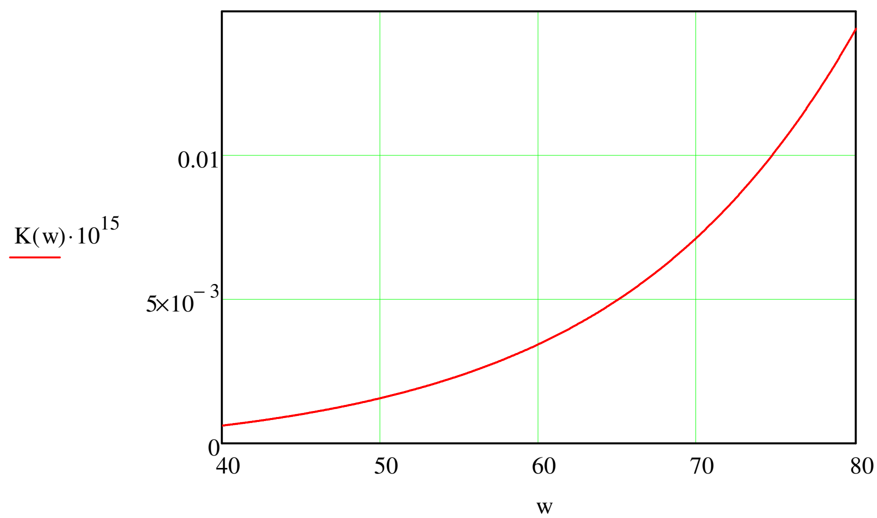


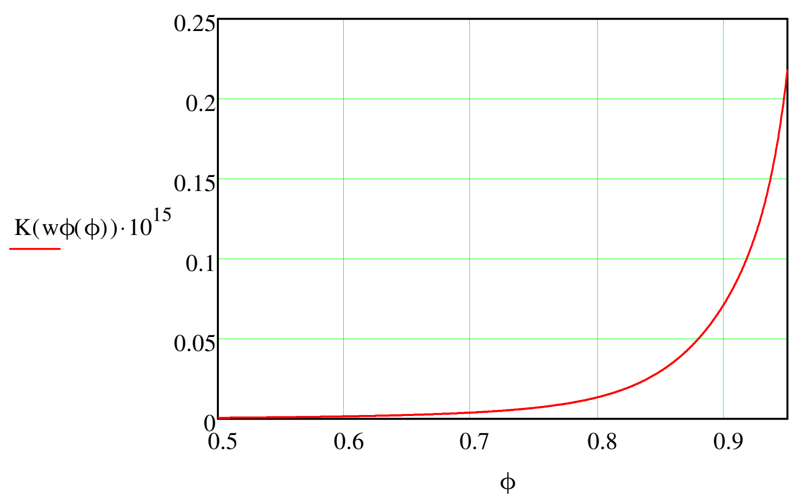
Liquid water permeability.

$$K(w) := e^{-39.2619+0.0704\cdot(w-73)-1.742\cdot 10^{-4}\cdot(w-73)^2-2.7953\cdot 10^{-6}\cdot(w-73)^3-1.1566\cdot 10^{-7}\cdot(w-73)^4+2.5969\cdot 10^{-9}\cdot(w-73)^5}$$




0.25
0.2
0.15
K(w)·10^15
0.1
0.05
0
80
90
100
110
120
130
w


0.01
K(w)·10^15
5×10^−3
0
40
50
60
70
80
w


0.25
0.2
0.15
K(wϕ(ϕ))·10^15
0.1
0.05
0
0.5
0.6
0.7
0.8
0.9
ϕ

Formula for p.sat(T).

Input from table:

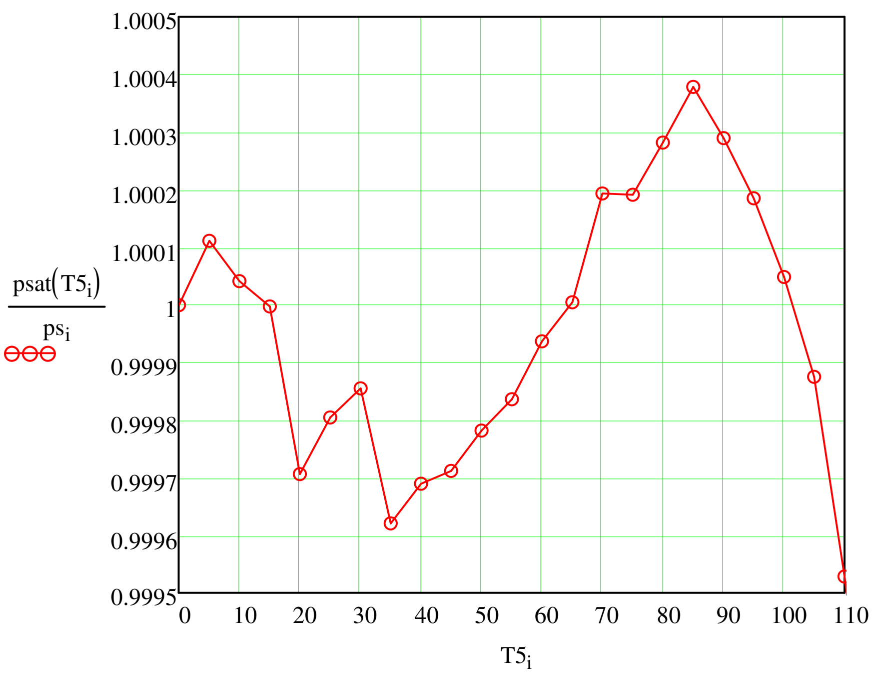
Tps :=

0	5	10	15	20	25	30	35	40	45	50	55	60
610.8	871.8	1227.0	1703.9	2337	3166	4241	5622	7375	9582	12335	15741	19920
65	70	75	80	85	90	95	100	105	110	115	120	125
25010	31160	38550	47360	57800	70110	84530	101330	120800	143270	169060	198540	232100

$i := 0..12$ $\quad T5_i := Tps_{(0,i)}$ $\quad ps_i := Tps_{(1,i)}$ $\quad T5_{i+13} := Tps_{(2,i)}$ $\quad ps_{i+13} := Tps_{(3,i)}$

$i := 0..25$

$$Epsat(T) := \frac{19.625 \cdot T}{270.1 + T} - \frac{0.00463 \cdot T^2}{142.15 + T} \qquad psat(T) := 610.8 \cdot e^{Epsat(T)}$$



Max relative error: 0.05% for 0<T<110.

Boltzmann solution of coupled nonlinear equations for moisture content w(s) and temperature T(s), s=x/root(4t). Benchmark test I, CEN. New version 2012. Control and accuracy of the solution

J. Claesson

Input data.

Initial and boundary conditions

$\phi in := 0.5 \quad Tin := 20 \qquad \phi b := 0.95 \quad Tb := 30$

Constants

$Tref := 20 \quad TKref := 273.15 + Tref \quad \rho w := 1000 \quad R := 8.314 \quad Mw := 0.018$

Sorption isotherm.

$$w\phi(\phi) := \frac{146}{\left[1 + \left(-8\cdot 10^{-8}\cdot\frac{R\cdot TKref\cdot\rho w}{Mw}\cdot\ln(\phi)\right)^{1.6}\right]^{0.375}}$$

$$\phi w(w) := e^{\frac{-Mw}{R\cdot TKref\cdot\rho w}\cdot 0.125\cdot 10^{8}\cdot\left[\left(\frac{146}{w}\right)^{\frac{1}{0.375}}-1\right]^{0.625}}$$

Initial and boundary moisture content:

$win := w\phi(\phi in) \quad wb := w\phi(\phi b) \qquad win = 42.922 \qquad wb = 129.021$

Water retention curve.

$$wPsuc(Psuc) := \frac{146}{\left[1 + \left(8\cdot 10^{-8}\cdot Psuc\right)^{1.6}\right]^{0.375}}$$

$$Psucw(w) := 0.125\cdot 10^{8}\cdot\left[\left(\frac{146}{w}\right)^{\frac{1}{0.375}} - 1\right]^{0.625}$$

Vapor diffusion coefficient.

$$\delta p(w) := \frac{Mw}{R\cdot TKref}\cdot\frac{26.1\cdot 10^{-6}}{200}\cdot\frac{1-\frac{w}{146}}{0.503\cdot\left(1-\frac{w}{146}\right)^{2}+0.497}$$

Liquid water permeability.

$$K(w) := e^{-39.2619+0.0704\cdot(w-73)-1.742\cdot 10^{-4}\cdot(w-73)^{2}-2.7953\cdot 10^{-6}\cdot(w-73)^{3}-1.1566\cdot 10^{-7}\cdot(w-73)^{4}+2.5969\cdot 10^{-9}\cdot(w-73)^{5}}$$

Formula for p.sat(T).

$$Epsat(T) := \frac{19.625\cdot T}{270.1+T} - \frac{0.00463\cdot T^{2}}{142.15+T} \qquad psat(T) := 610.8\cdot e^{Epsat(T)}$$

Heat of evaporation.

$$Lev(T) := 10^{6}\cdot\left[2.5016 - \frac{T}{422} - \left(\frac{T}{342}\right)^{4}\right]$$

Heat content (enthalpy) of water and water vapor.

$hl(T) := 4184\cdot T \qquad hv(T) := hl(T) + Lev(T)$

Thermal conductivity.

$$\lambda(w) := 1.5 + 0.0158 \cdot w$$

Water vapor content

$Vp := 0.146$ (pore volume)

$$wvap(w,T) := \left(Vp - \frac{w}{\rho w}\right) \cdot \phi w(w) \cdot \frac{Mw \cdot psat(T)}{R \cdot (273.15 + T)}$$

Total energy (enthalpy)

$$h(w,T) := 1.824 \cdot 10^{6} \cdot T + hl(T) \cdot w + Lev(T) \cdot wvap(w,T)$$ (only used for control)

Capacity functions (derivatives of h(w,T)).

$$hw(w,T) := hl(T) + Lev(T) \cdot \frac{Mw \cdot psat(T)}{R \cdot (273.15 + T)} \cdot \left[-\frac{\phi w(w)}{\rho w} + \left(Vp - \frac{w}{\rho w}\right) \cdot \left(\frac{d}{dw} \phi w(w)\right) \right]$$

$$hT(w,T) := 1.824 \cdot 10^{6} + 4184 \cdot w + \left(Vp - \frac{w}{\rho w}\right) \cdot \frac{\phi w(w) \cdot Mw}{R} \cdot \frac{d}{dT}\left(\frac{Lev(T) \cdot psat(T)}{273.15 + T}\right)$$

Auxilliary functions.

Functions for flow coefficients:

$$Kw(w,T) := K(w) \cdot \frac{d}{dw}(-Psucw(w)) + psat(T) \cdot \delta p(w) \cdot \frac{d}{dw}(\phi w(w))$$

$$\lambda w(w,T) := -hl(T) \cdot K(w) \cdot \frac{d}{dw}(Psucw(w)) + hv(T) \cdot psat(T) \cdot \delta p(w) \cdot \frac{d}{dw}(\phi w(w))$$

$$KT(w,T) := \delta p(w) \cdot \phi w(w) \cdot \frac{d}{dT} psat(T) \qquad \lambda T(w,T) := \lambda(w) + hv(T) \cdot \delta p(w) \cdot \phi w(w) \cdot \frac{d}{dT} psat(T)$$

Determinant and R-functions:

$$DetK\lambda(w,T) := Kw(w,T) \cdot \lambda T(w,T) - KT(w,T) \cdot \lambda w(w,T)$$

$$R_1(w,T) := \frac{\lambda T(w,T)}{DetK\lambda(w,T)} \qquad R_2(w,T) := \frac{-KT(w,T)}{DetK\lambda(w,T)} \qquad R_3(w,T) := \frac{-\lambda w(w,T)}{DetK\lambda(w,T)}$$

$$R_4(w,T) := \frac{Kw(w,T)}{DetK\lambda(w,T)} \qquad R_5(w,T) := \frac{\lambda T(w,T)}{DetK\lambda(w,T)} \qquad R_6(w,T) := \frac{-KT(w,T)}{DetK\lambda(w,T)}$$

$$R_7(w,T) := \frac{hw(w,T) \cdot \lambda T(w,T) - hT(w,T) \cdot \lambda w(w,T)}{DetK\lambda(w,T)} \qquad R_8(w,T) := \frac{-hw(w,T) \cdot KT(w,T) + hT(w,T) \cdot Kw(w,T)}{DetK\lambda(w,T)}$$

Calculations.

Estimates of moisture and temperature penetration depths.

$$Lpenw := \sqrt{Kw(wb,Tb)} \qquad LpenT := \sqrt{\frac{\lambda T(wb,Tb)}{hT(wb,Tb)}} \qquad Lpenw = 8.469 \times 10^{-6} \qquad LpenT = 1.224 \times 10^{-3}$$

Given

$$\frac{d}{ds}w(s) = -\left(R_1(w(s),T(s))\cdot g(s) + R_2(w(s),T(s))\cdot q(s)\right)$$

$w(0) = wb$ 　 $wb = 129.021$

$Tb = 30$

$$\frac{d}{ds}T(s) = -\left(R_3(w(s),T(s))\cdot g(s) + R_4(w(s),T(s))\cdot q(s)\right)$$

$T(0) = Tb$

$Tin = 20$

$$\frac{d}{ds}g(s) = -2\cdot s\cdot\left(R_5(w(s),T(s))\cdot g(s) + R_6(w(s),T(s))\cdot q(s)\right)$$

$g(0) = 5.932\cdot 10^{-4}$ 　 $win = 42.922$

$$\frac{d}{ds}q(s) = -2\cdot s\cdot\left(R_7(w(s),T(s))\cdot g(s) + R_8(w(s),T(s))\cdot q(s)\right)$$

$q(0) = 2.369\cdot 10^{4}$

$smax := 0.003$ 　 $J := 50000$

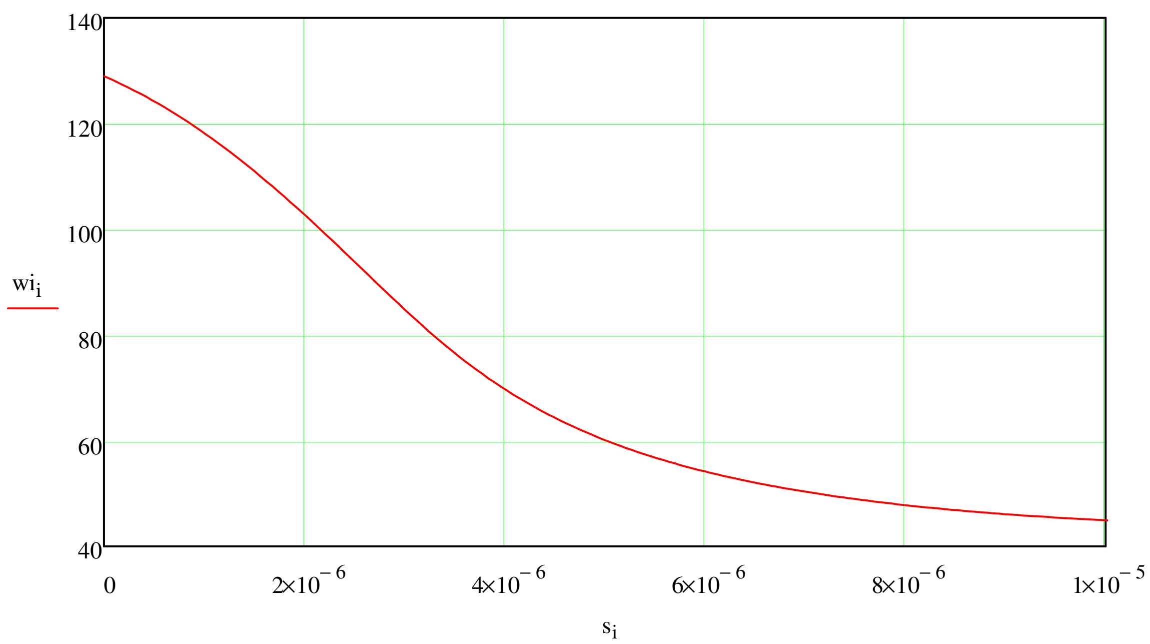
$$\begin{pmatrix} w \\ T \\ g \\ q \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} w \\ T \\ g \\ q \end{pmatrix}, s, smax, J\right]$$

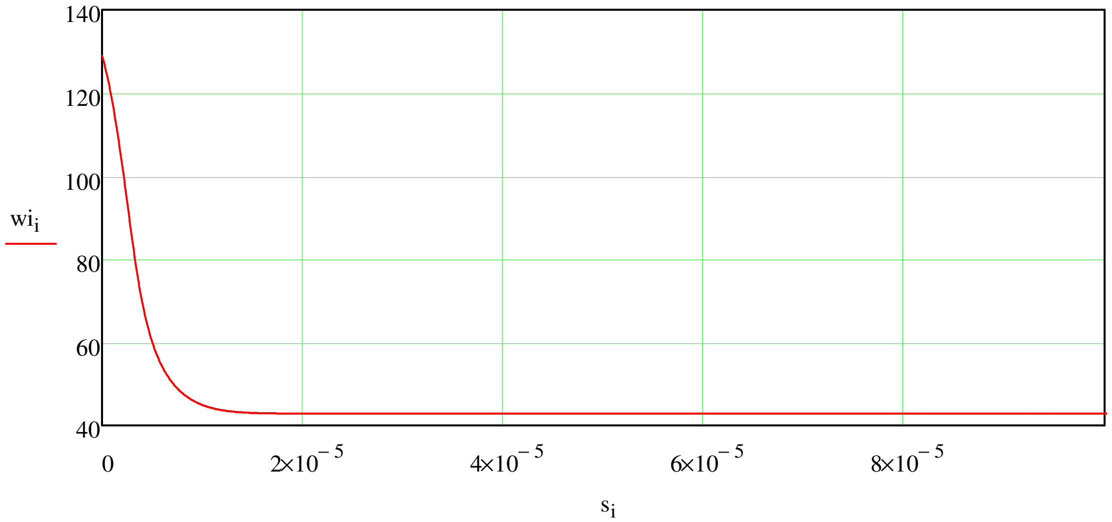
$i := 0..J$ 　 $s_i := \frac{i}{J}\cdot smax$

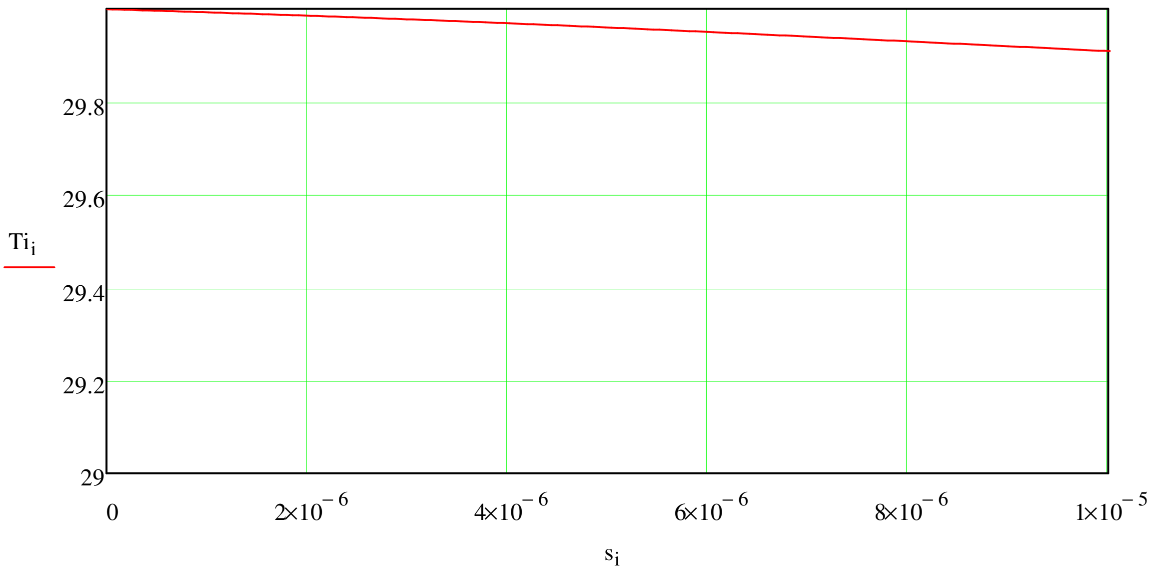
$smax = 3\times 10^{-3}$

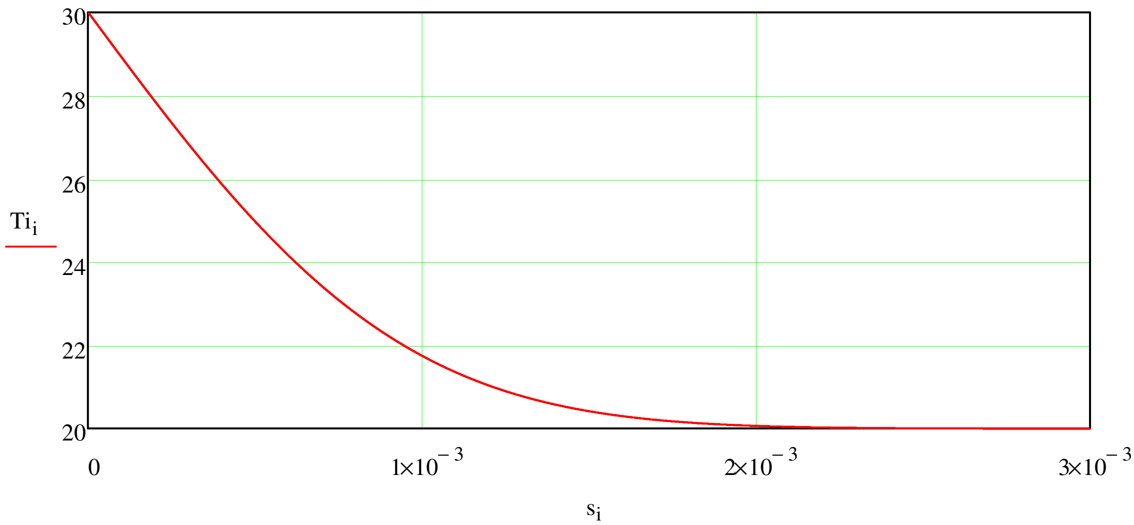
$w(0) = 129.021$ 　 $T(0) = 30$ 　 $w(smax) = 42.924$ 　 $T(smax) = 20.002$

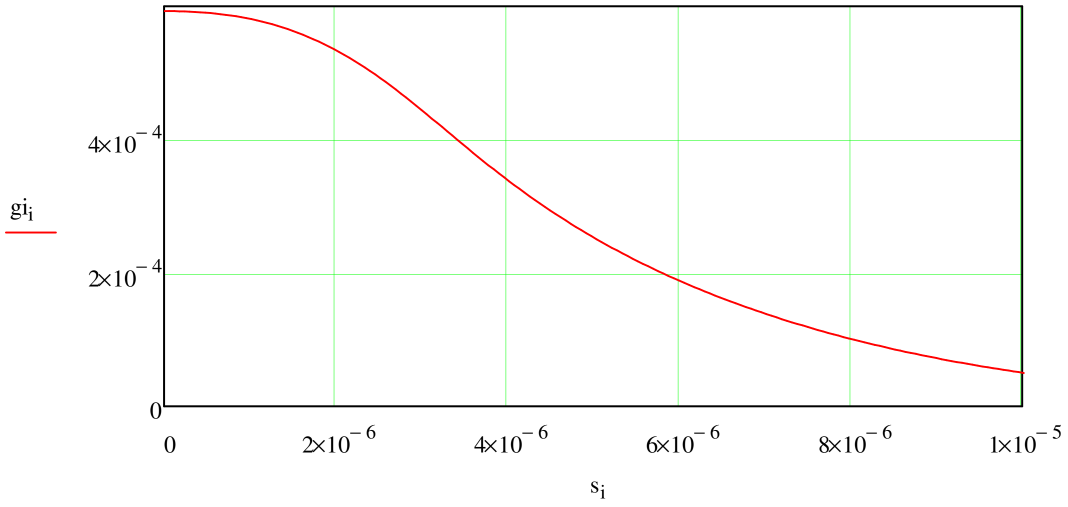
$i := 0..J$ 　 $wi_i := w(s_i)$ 　 $Ti_i := T(s_i)$ 　 $gi_i := g(s_i)$ 　 $qi_i := q(s_i)$



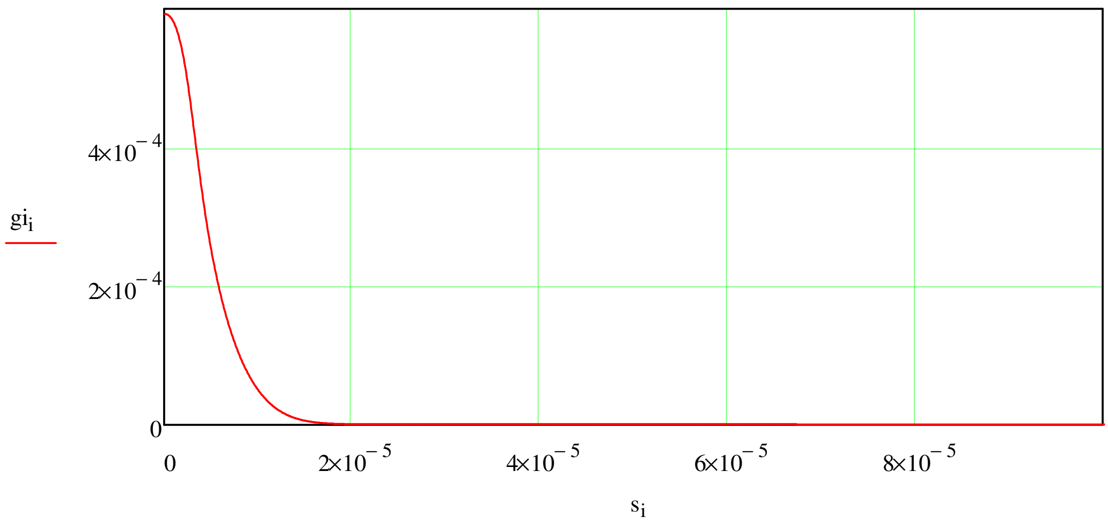

140
120
100
80
60
40
wi_i
0
2×10⁻⁵
4×10⁻⁵
6×10⁻⁵
8×10⁻⁵
s_i


29.8
29.6
29.4
29.2
29
Ti_i
0
2×10⁻⁶
4×10⁻⁶
6×10⁻⁶
8×10⁻⁶
1×10⁻⁵
s_i

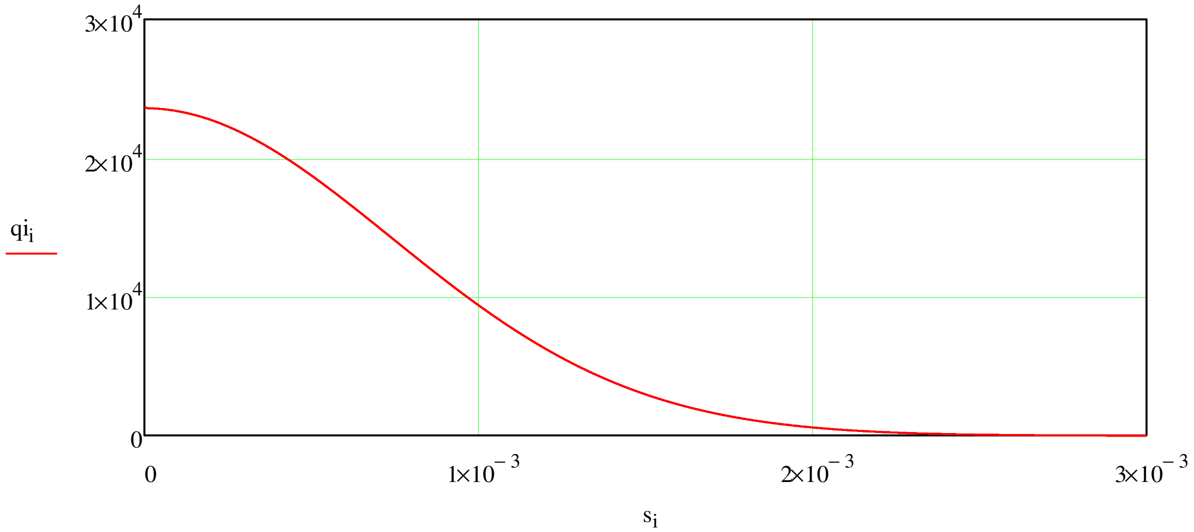

30
28
26
24
22
20
Ti_i
0
1×10⁻³
2×10⁻³
3×10⁻³
s_i



4×10^−4
2×10^−4
0
gi_i
0
2×10^−6
4×10^−6
6×10^−6
8×10^−6
1×10^−5
s_i



4×10^−4
2×10^−4
0
gi_i
0
2×10^−5
4×10^−5
6×10^−5
8×10^−5
s_i



3×10^4
2×10^4
1×10^4
0
qi_i
0
1×10^−3
2×10^−3
3×10^−3
s_i

Control of solution.

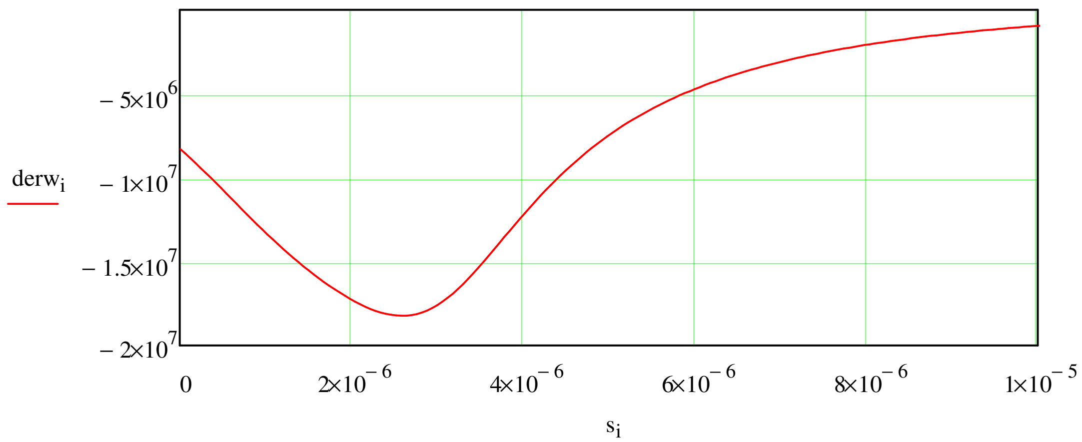
Derivatives of w, T, g and q

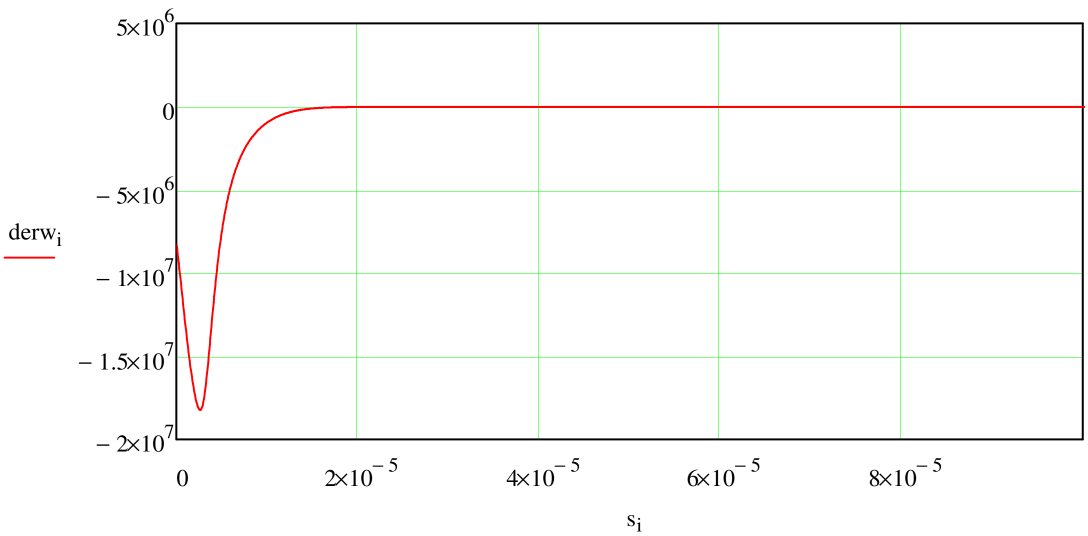
$$i := 0..J \qquad Det_i := Kw(wi_i, Ti_i)\cdot\lambda T(wi_i, Ti_i) - KT(wi_i, Ti_i)\cdot\lambda w(wi_i, Ti_i)$$

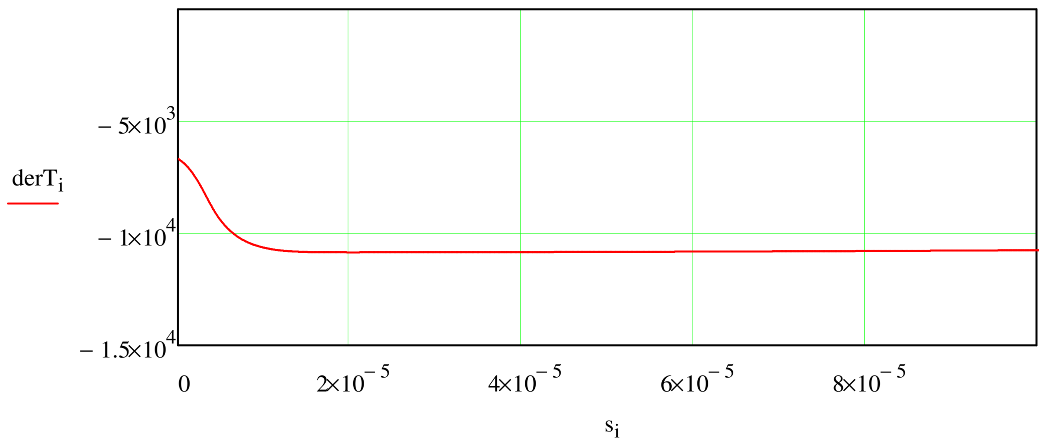
$$derw_i := -\frac{\lambda T(wi_i, Ti_i)\cdot gi_i - KT(wi_i, Ti_i)\cdot qi_i}{Det_i} \qquad derT_i := -\frac{-\lambda w(wi_i, Ti_i)\cdot gi_i + Kw(wi_i, Ti_i)\cdot qi_i}{Det_i}$$

$$derg_i := -2\cdot s_i\cdot\frac{\lambda T(wi_i, Ti_i)\cdot gi_i - KT(wi_i, Ti_i)\cdot qi_i}{Det_i} \qquad R7_i := hw(wi_i, Ti_i)\cdot\lambda T(wi_i, Ti_i) - hT(wi_i, Ti_i)\cdot\lambda w(wi_i, Ti_i)$$

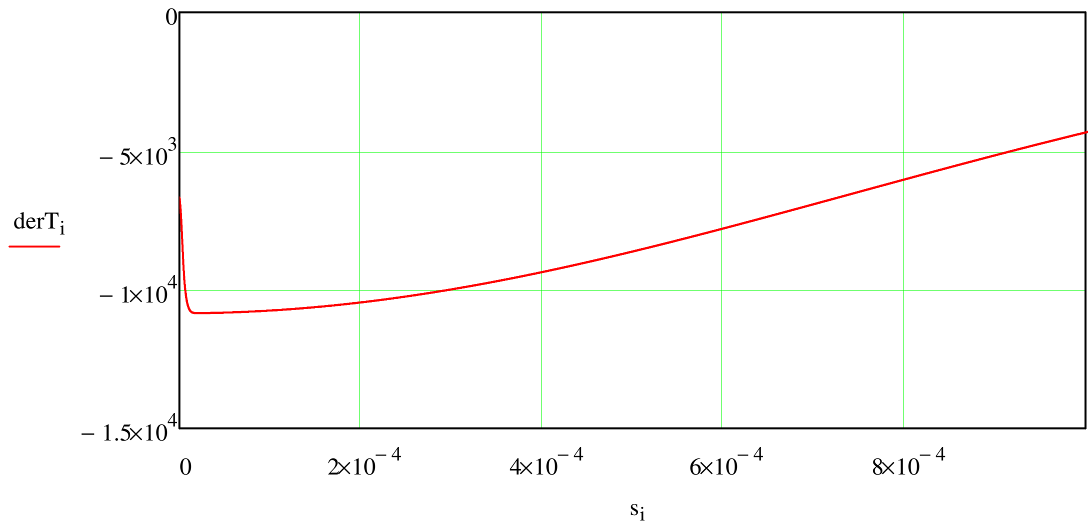
$$derq_i := -2\cdot s_i\cdot\frac{R7_i\cdot gi_i - \left(hw(wi_i, Ti_i)\cdot KT(wi_i, Ti_i) - hT(wi_i, Ti_i)\cdot Kw(wi_i, Ti_i)\right)\cdot qi_i}{Det_i}$$



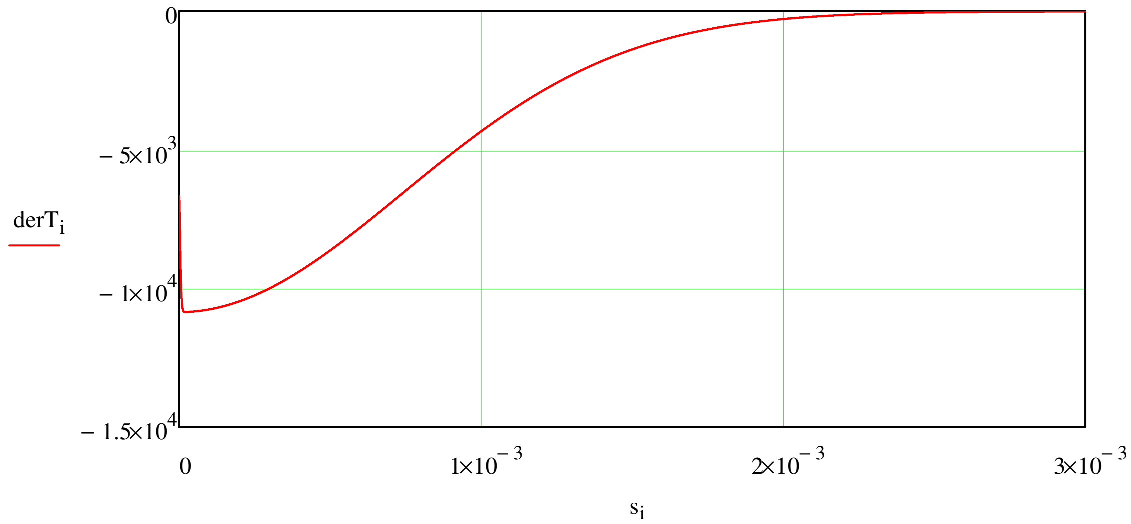




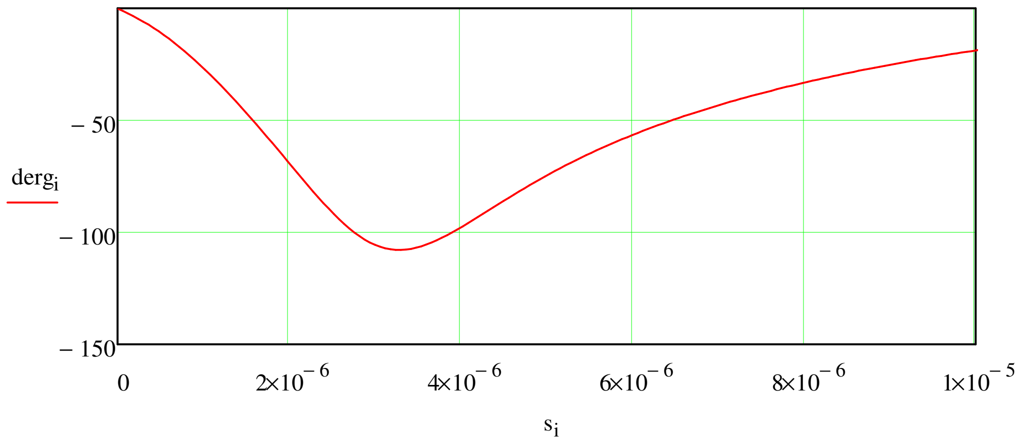
derT_i
− 5×10^3
− 1×10^4
− 1.5×10^4
0
2×10^−5
4×10^−5
6×10^−5
8×10^−5
s_i



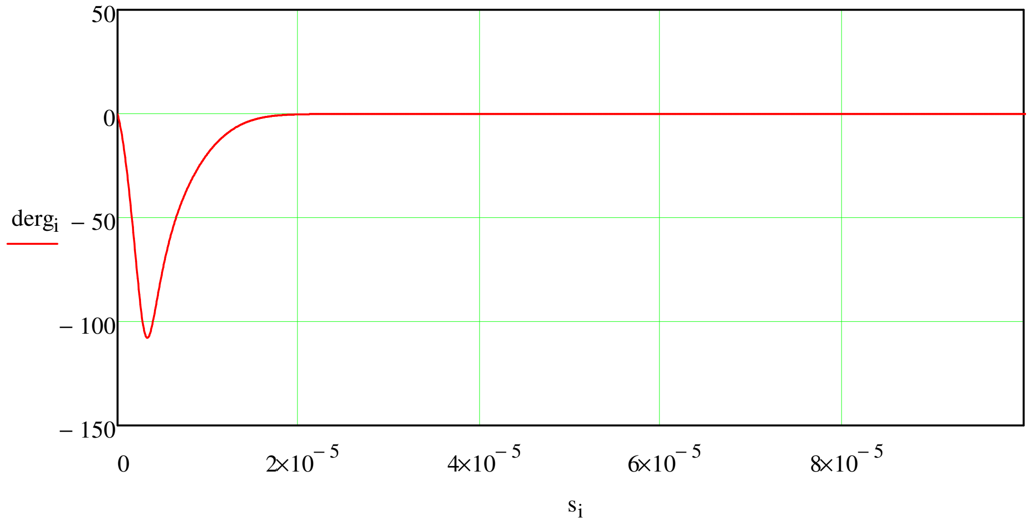
derT_i
0
− 5×10^3
− 1×10^4
− 1.5×10^4
0
2×10^−4
4×10^−4
6×10^−4
8×10^−4
s_i



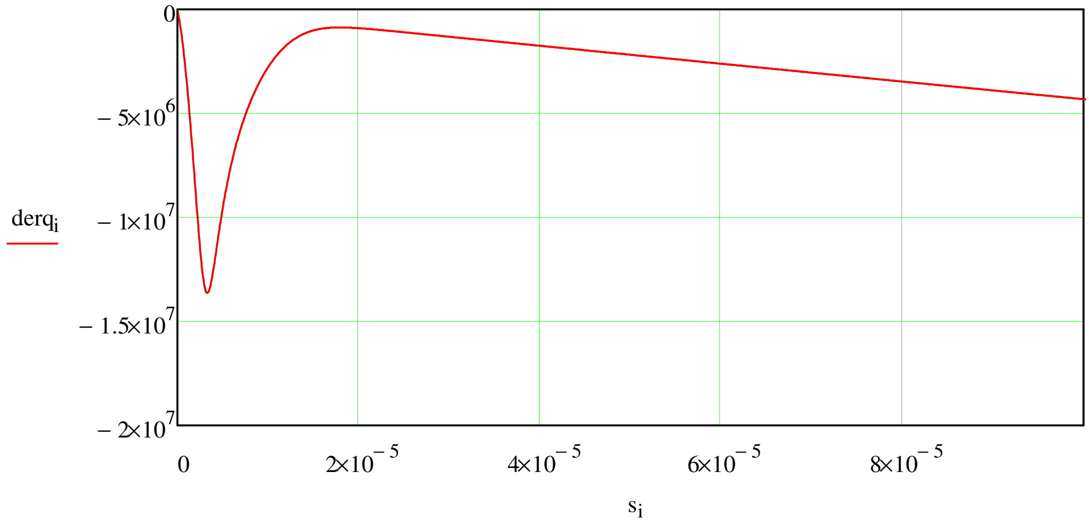
derT_i
0
− 5×10^3
− 1×10^4
− 1.5×10^4
0
1×10^−3
2×10^−3
3×10^−3
s_i



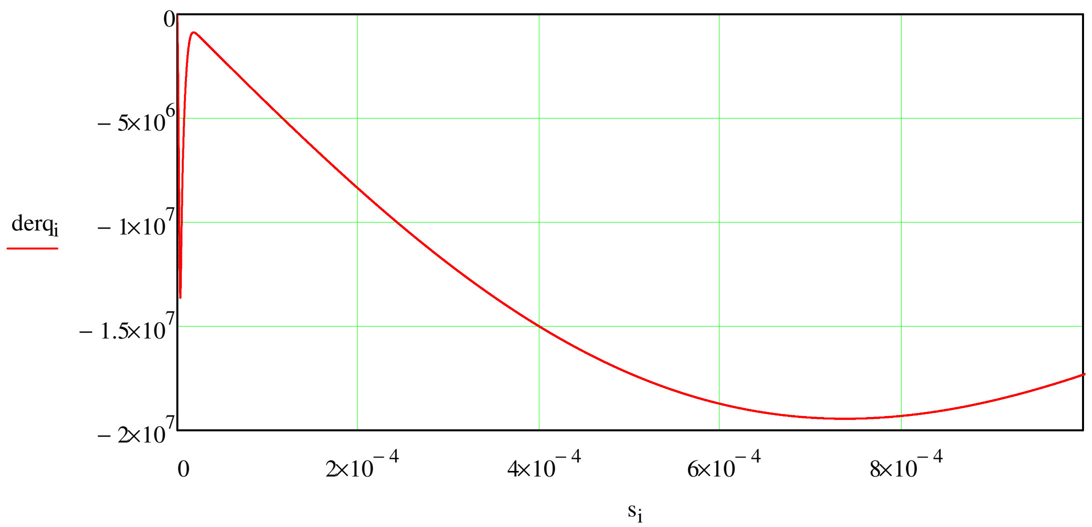
− 50
− 100
− 150
$derg_i$
0
2×10^{-6}
4×10^{-6}
6×10^{-6}
8×10^{-6}
1×10^{-5}
s_i

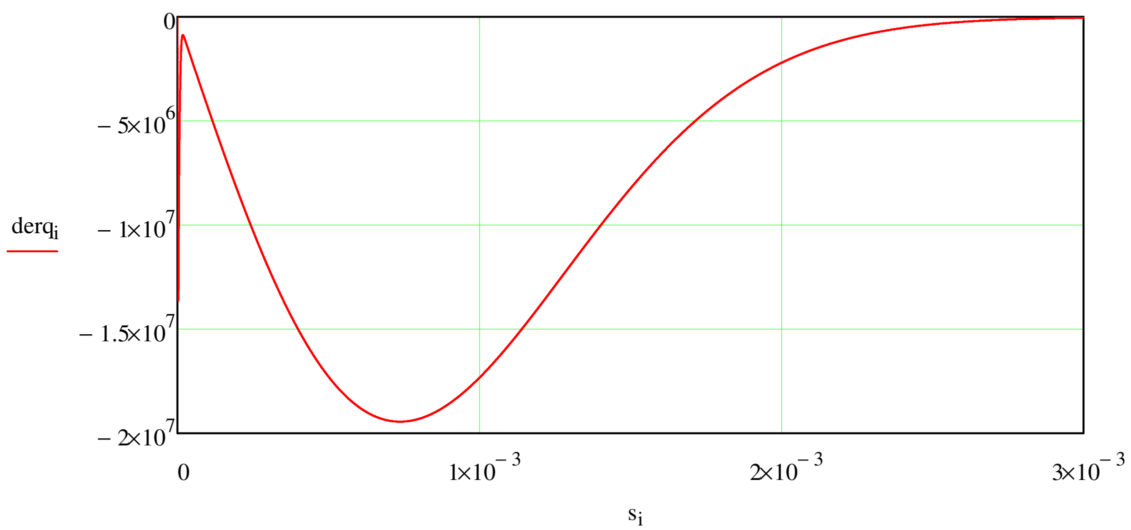


50
0
− 50
− 100
− 150
$derg_i$
0
2×10^{-5}
4×10^{-5}
6×10^{-5}
8×10^{-5}
s_i



0
-5×10^{6}
-1×10^{7}
-1.5×10^{7}
-2×10^{7}
$derq_i$
0
2×10^{-5}
4×10^{-5}
6×10^{-5}
8×10^{-5}
s_i





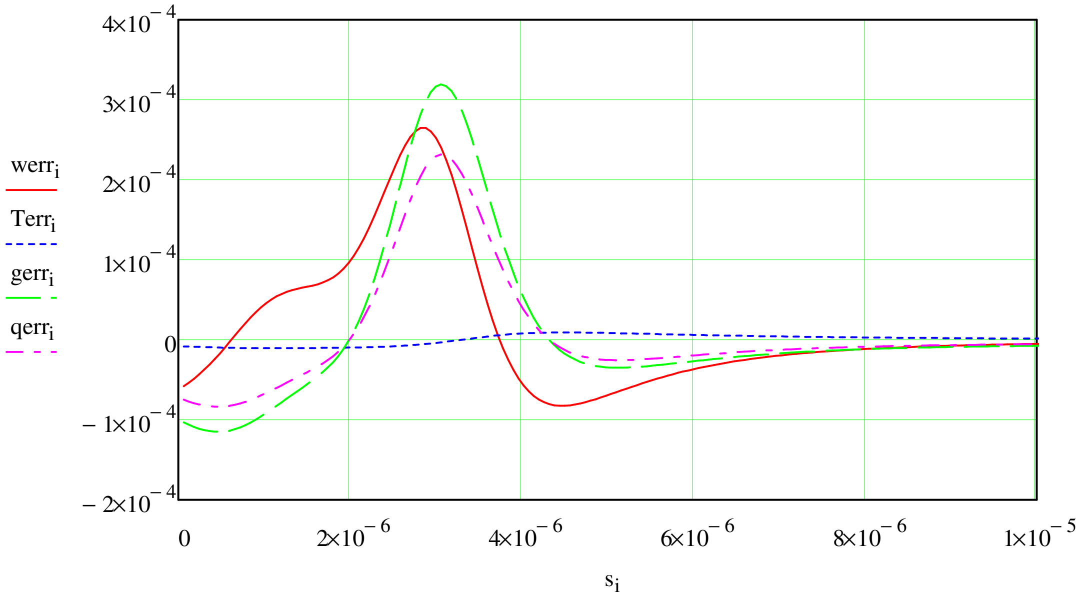
Errors for the four coupled ordinary differential equations:

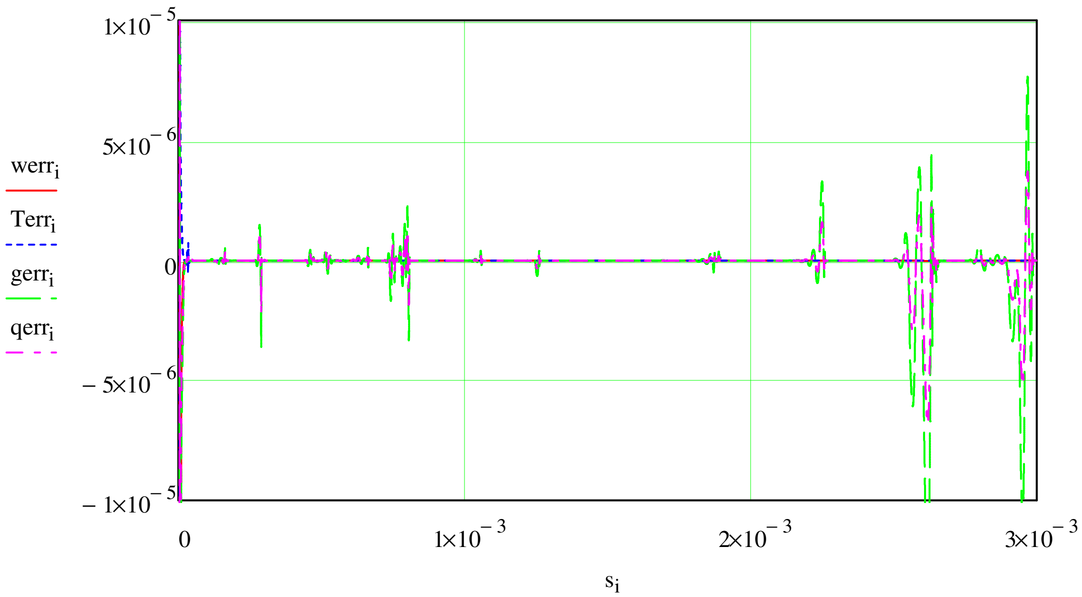
$i := 1 .. J - 1$

$$werr_i := \frac{\dfrac{wi_{i+1} - wi_{i-1}}{s_{i+1} - s_{i-1}} - derw_i}{1.8 \cdot 10^7} \qquad Terr_i := \frac{\dfrac{Ti_{i+1} - Ti_{i-1}}{s_{i+1} - s_{i-1}} - derT_i}{11000}$$

$$gerr_i := \frac{\dfrac{gi_{i+1} - gi_{i-1}}{s_{i+1} - s_{i-1}} - derg_i}{110} \qquad qerr_i := \frac{\dfrac{qi_{i+1} - qi_{i-1}}{s_{i+1} - s_{i-1}} - derq_i}{1.9 \cdot 10^7}$$

Note: maxima derw=1.8*10^7, derT=11000, derg=110, derq=1.9*10^7



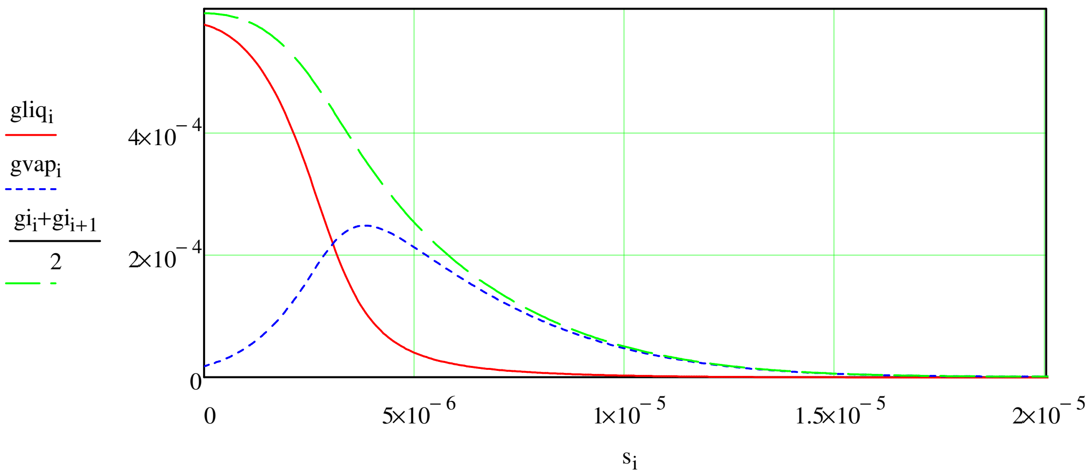


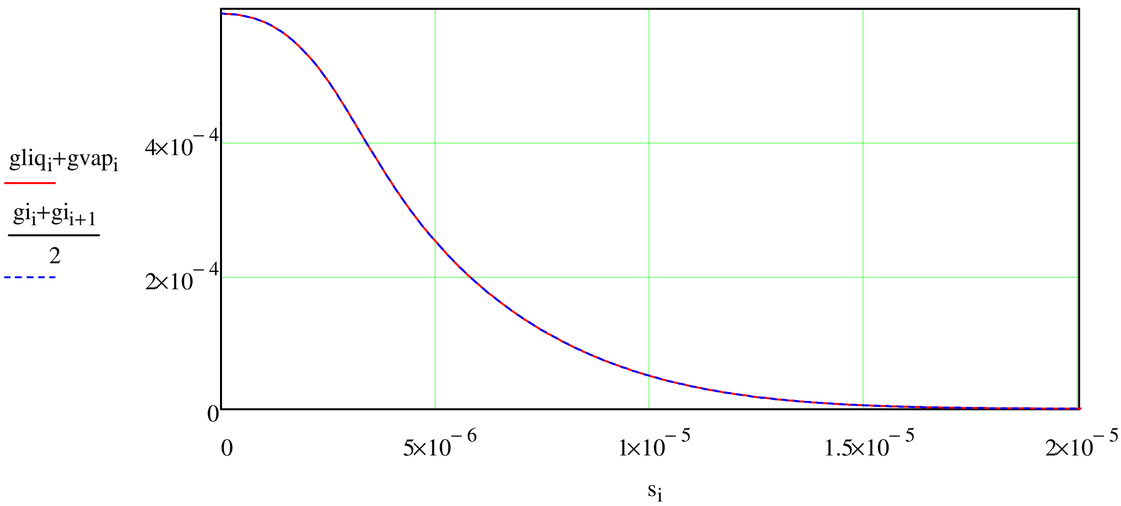
Conclusion: Errors below 0.03%.

Vapor and liquid moisture flux.

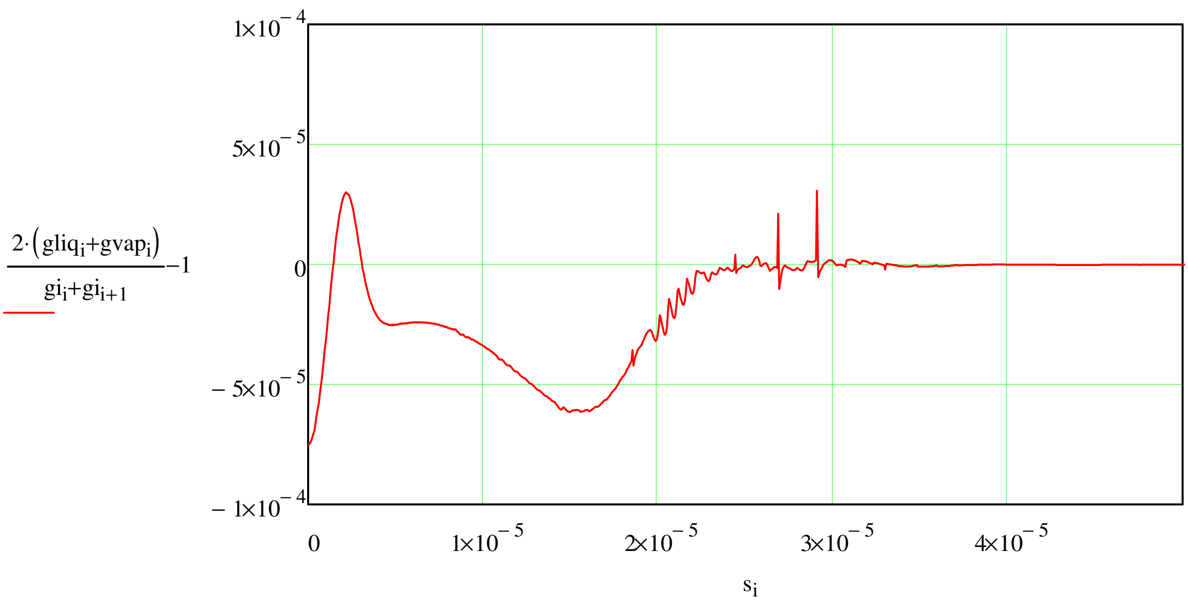
$i := 0 .. J$ $\qquad Psuc_i := Psucw(wi_i)$ $\qquad p_i := \phi w(wi_i) \cdot psat(Ti_i)$

$i := 0 .. J - 1$ $\qquad gliq_i := K\left(\frac{wi_i + wi_{i+1}}{2}\right) \cdot \frac{Psuc_{i+1} - Psuc_i}{s_{i+1} - s_i}$ $\qquad gvap_i := -\delta p\left(\frac{wi_i + wi_{i+1}}{2}\right) \cdot \frac{p_{i+1} - p_i}{s_{i+1} - s_i}$



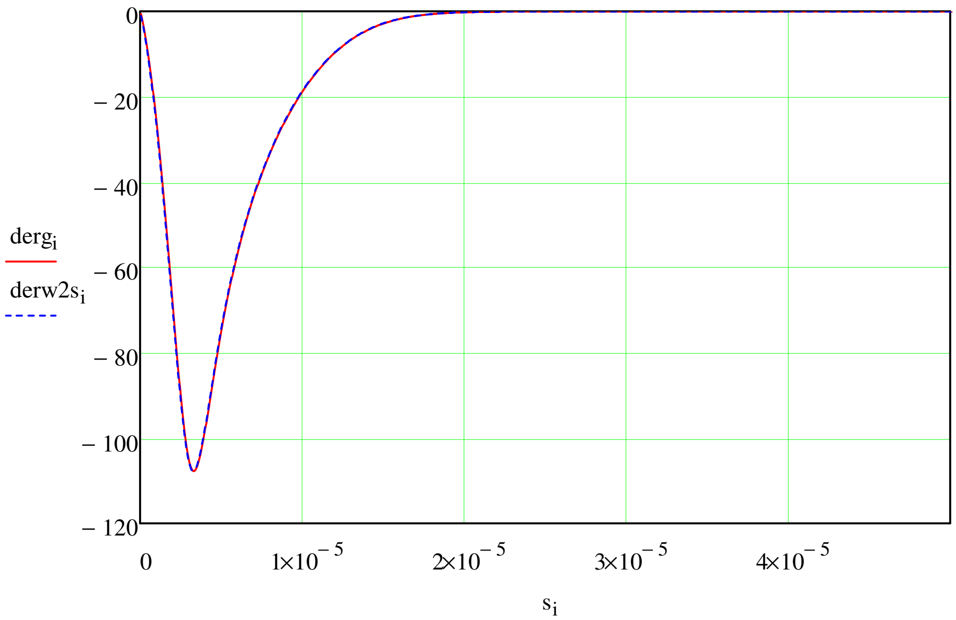


$$gliq_{J-1} + gvap_{J-1} = 1.812 \times 10^{-10}$$

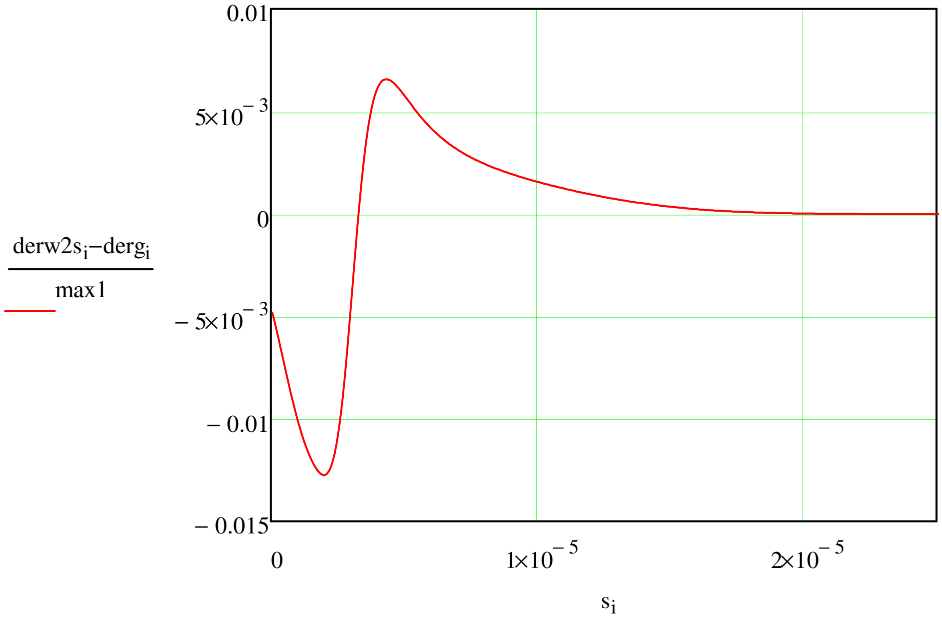


Moisture balance.

$i := 1 .. J - 1$ $\qquad$ $derg_i := \dfrac{gi_i - gi_{i-1}}{0.5 \cdot (s_{i+1} - s_{i-1})}$ $\qquad$ $derw2s_i := 2 \cdot s_i \cdot \dfrac{wi_{i+1} - wi_{i-1}}{s_{i+1} - s_{i-1}}$

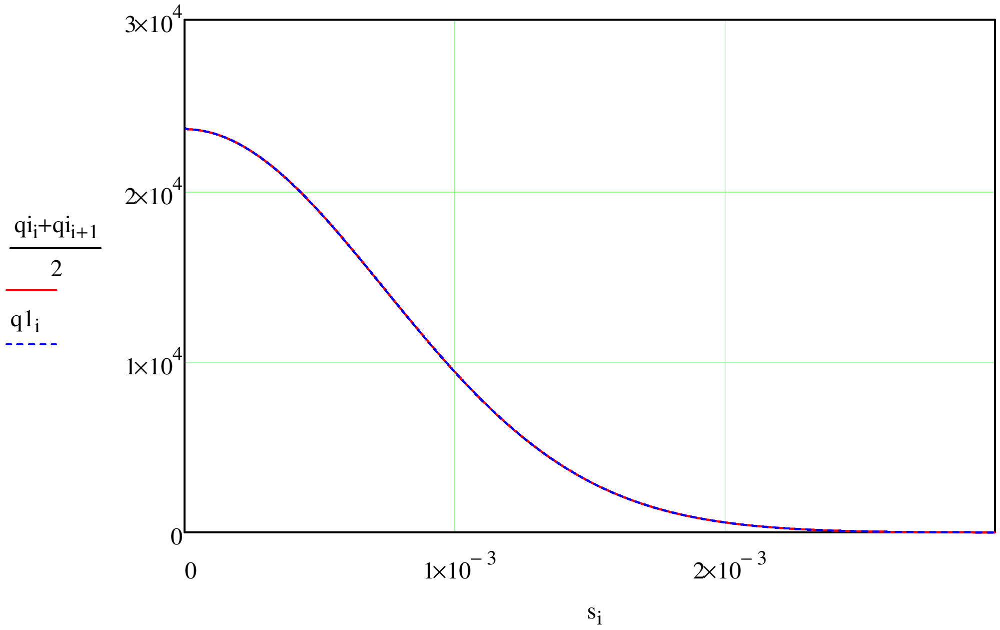


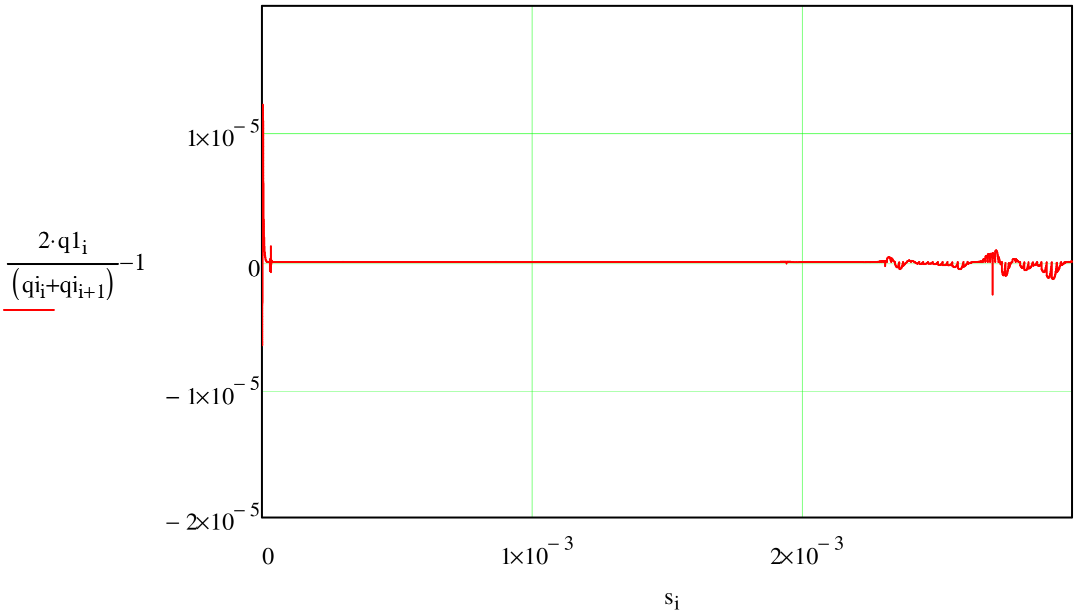
$max1 := max(-derg)$ $\qquad$ $max1 = 107.685$

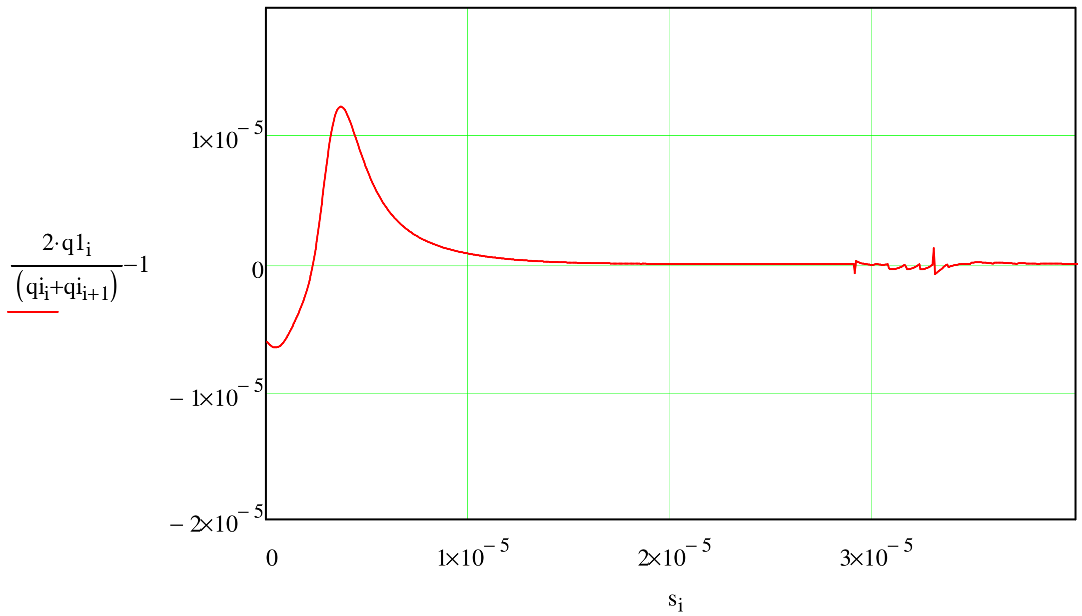


Enthalpy flux.

$$q1_i := -\lambda\left(\frac{wi_i + wi_{i+1}}{2}\right)\cdot\frac{Ti_{i+1} - Ti_i}{s_{i+1} - s_i} + hl\left(\frac{Ti_i + Ti_{i+1}}{2}\right)\cdot gliq_i + hv\left(\frac{Ti_i + Ti_{i+1}}{2}\right)\cdot gvap_i$$



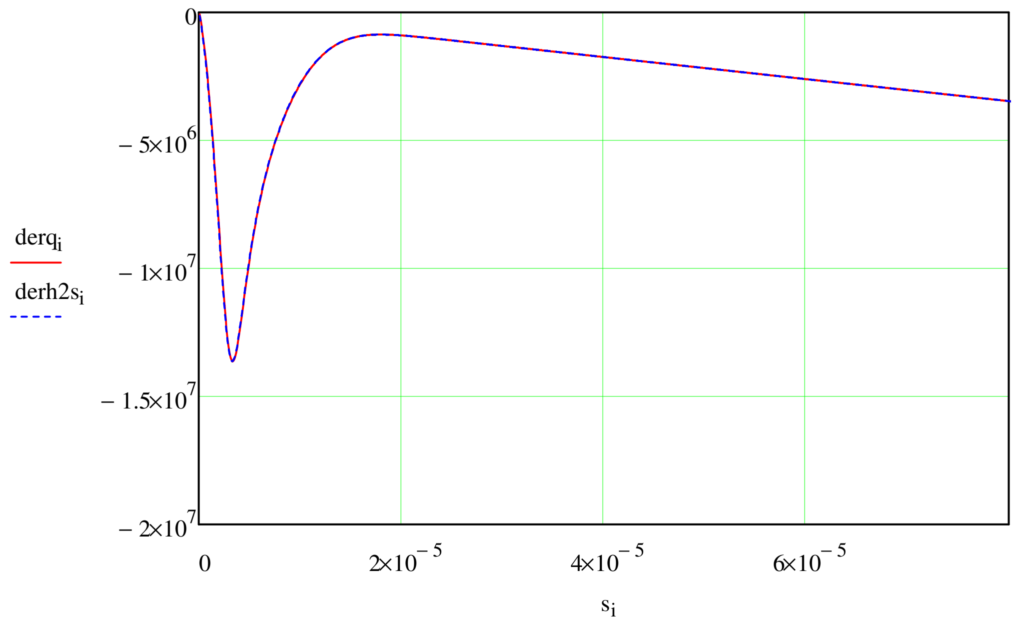




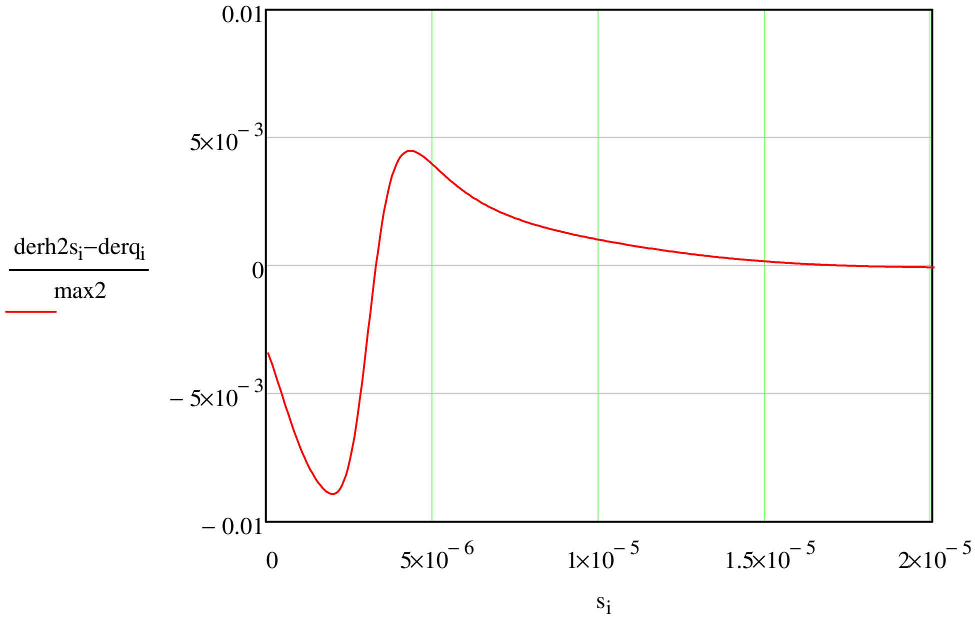
Energy (enthalpy) balance.

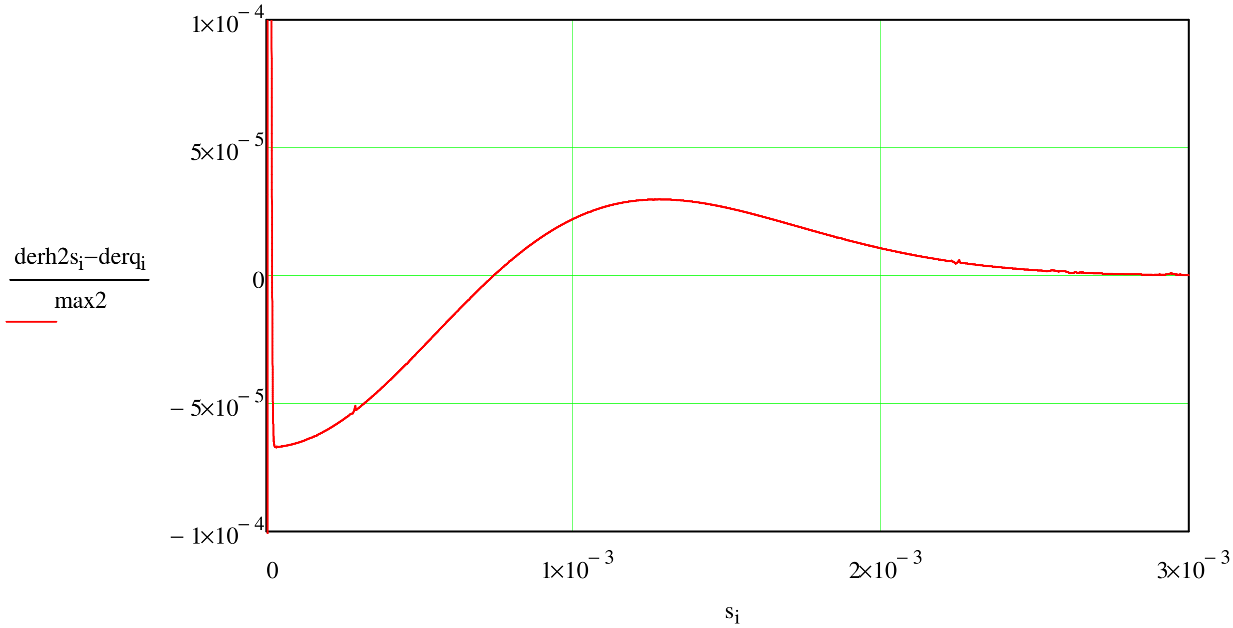
$i := 0..J$ $\qquad$ $h1_i := h(wi_i, Ti_i)$

$i := 1..J-1$ $\qquad$ $derq_i := \frac{qi_i - qi_{i-1}}{0.5 \cdot (s_{i+1} - s_{i-1})}$ $\qquad$ $derh2s_i := 2 \cdot s_i \cdot \frac{h1_{i+1} - h1_{i-1}}{s_{i+1} - s_{i-1}}$



$max2 := max(-derq)$ $\qquad$ $max2 = 1.943 \times 10^7$





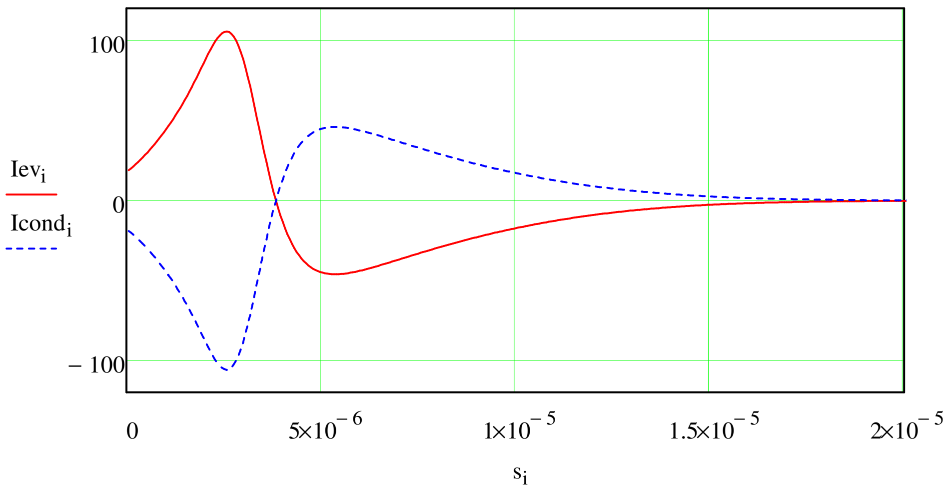
Evaporation and condensation of moisture.

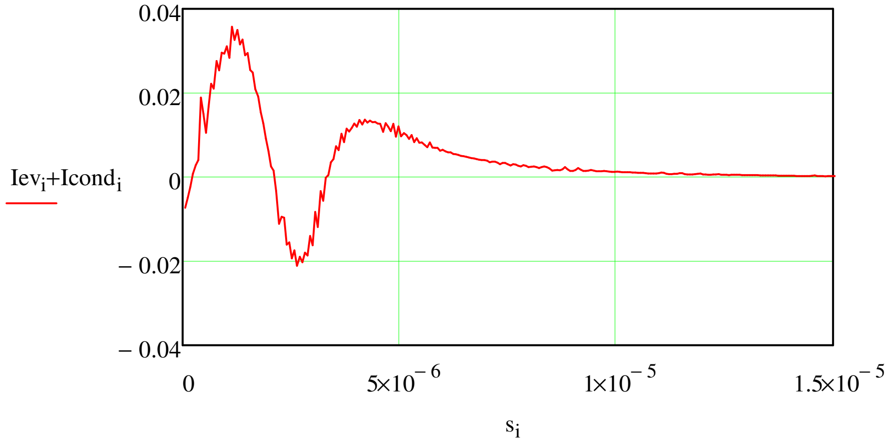
$i := 0..J \qquad wv_i := wvap(wi_i, Ti_i) \qquad wl_i := wi_i - wv_i$

$i := 1..J-1$

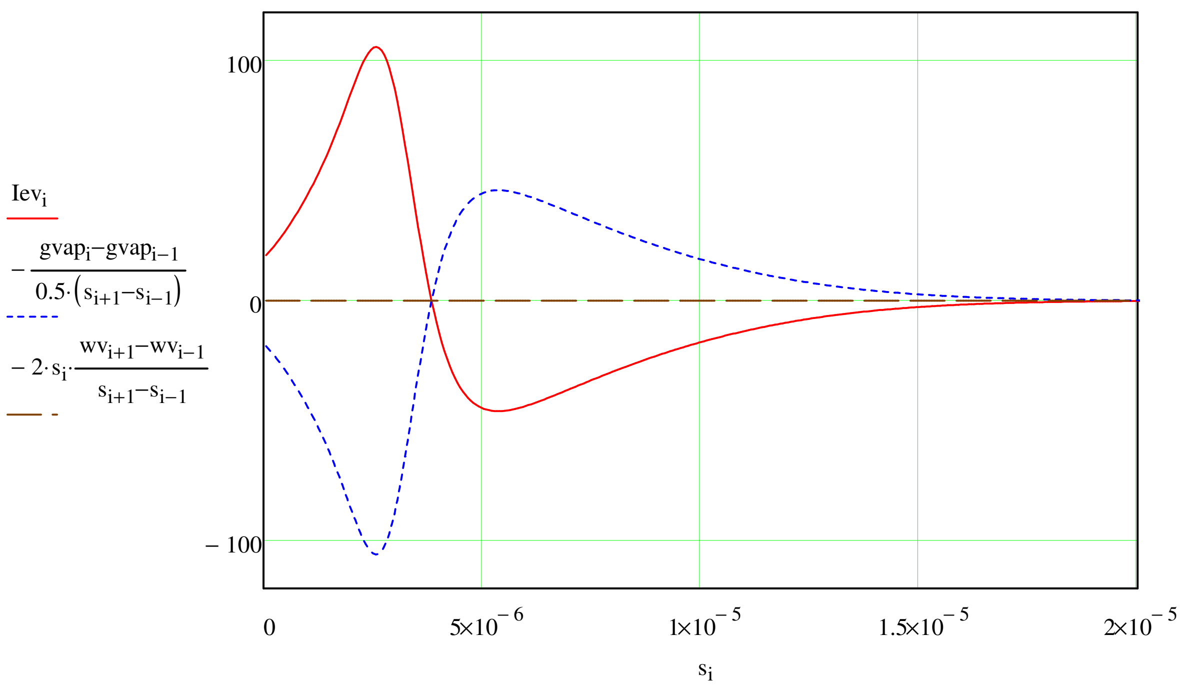
$$Iev_i := -2 \cdot s_i \cdot \frac{wv_{i+1} - wv_{i-1}}{s_{i+1} - s_{i-1}} + \frac{gvap_i - gvap_{i-1}}{0.5 \cdot (s_{i+1} - s_{i-1})}$$

$$Icond_i := -2 \cdot s_i \cdot \frac{wl_{i+1} - wl_{i-1}}{s_{i+1} - s_{i-1}} + \frac{gliq_i - gliq_{i-1}}{0.5 \cdot (s_{i+1} - s_{i-1})}$$

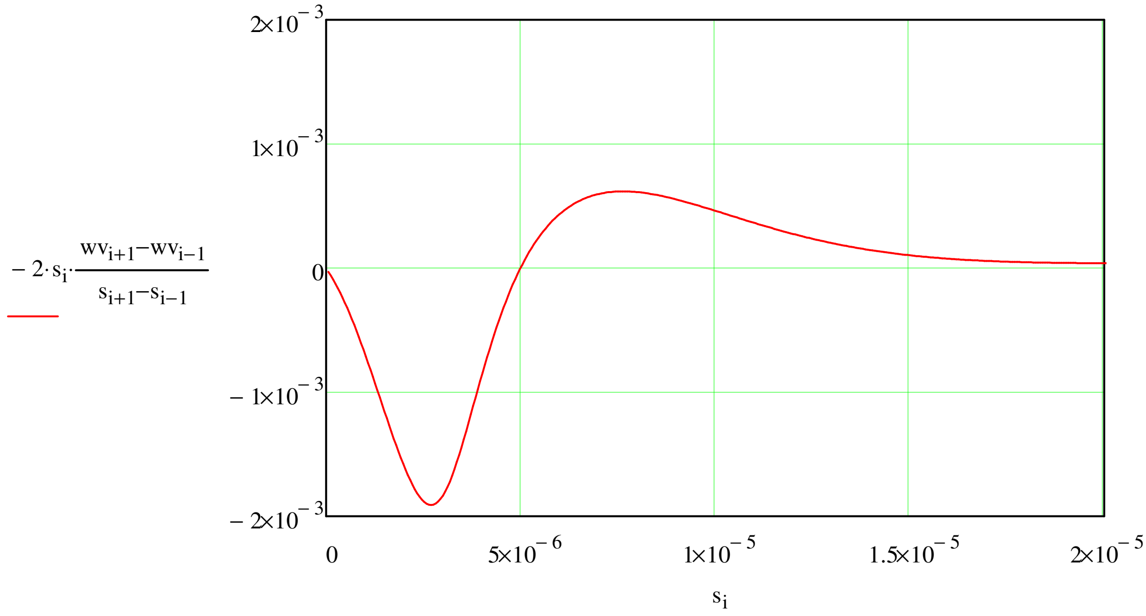




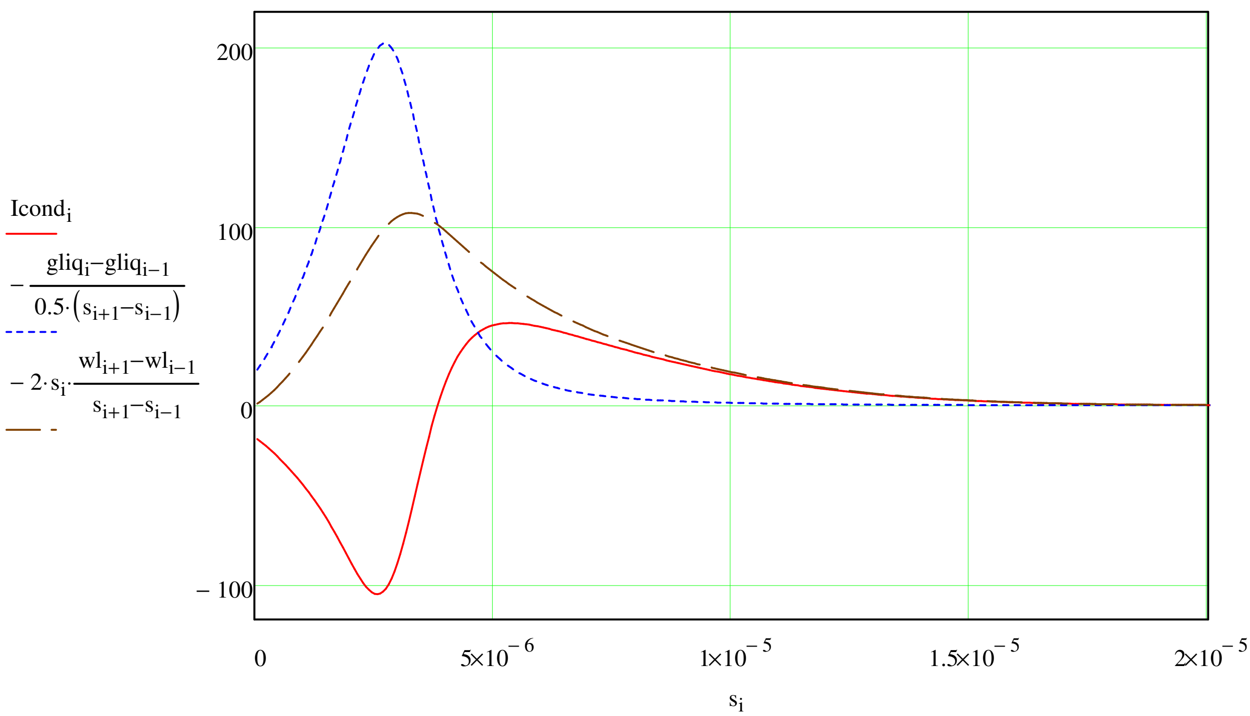
Vapor balance: Evaporation + net influx = rate of increase for water vapor content.



Rate of increase for water vapor content.



Balance for liquid water: Condensation + net influx = rate of increase for liquid water content.



Curves for presentation.

$i := 1 .. J - 1$

$$Iliqconv_i := -\frac{gliq_i - gliq_{i-1}}{0.5 \cdot (s_{i+1} - s_{i-1})}$$

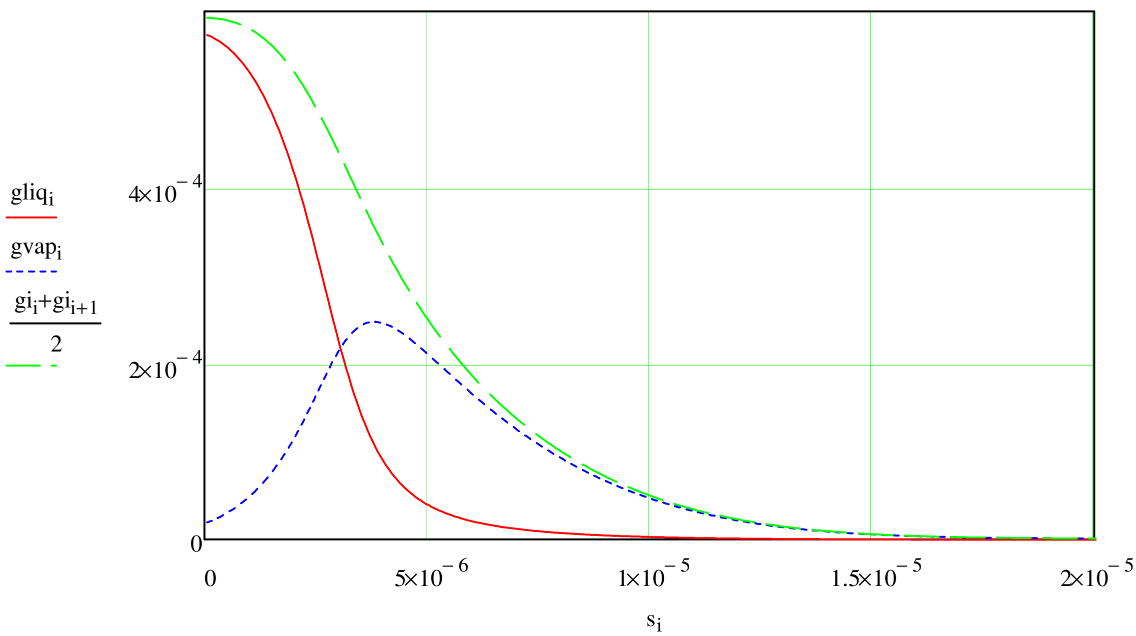
$$Iliqinc_i := -2 \cdot s_i \cdot \frac{wl_{i+1} - wl_{i-1}}{s_{i+1} - s_{i-1}}$$

$$Ivapconv_i := -\frac{gvap_i - gvap_{i-1}}{0.5 \cdot (s_{i+1} - s_{i-1})}$$

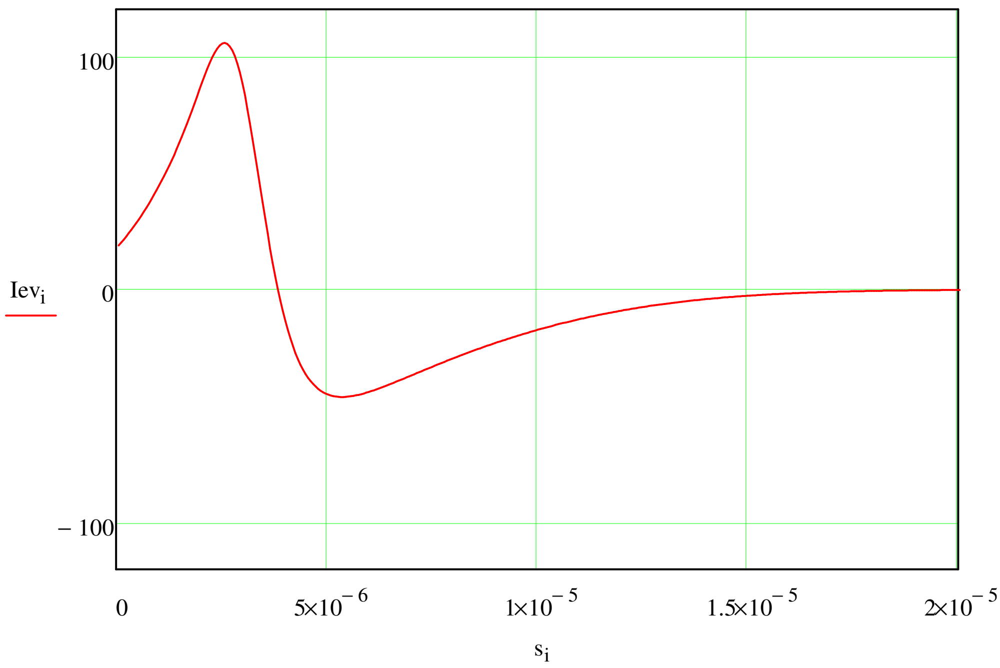
$$Ivap_i := -2 \cdot s_i \cdot \frac{wv_{i+1} - wv_{i-1}}{s_{i+1} - s_{i-1}}$$

Liquid and vapor flux of water

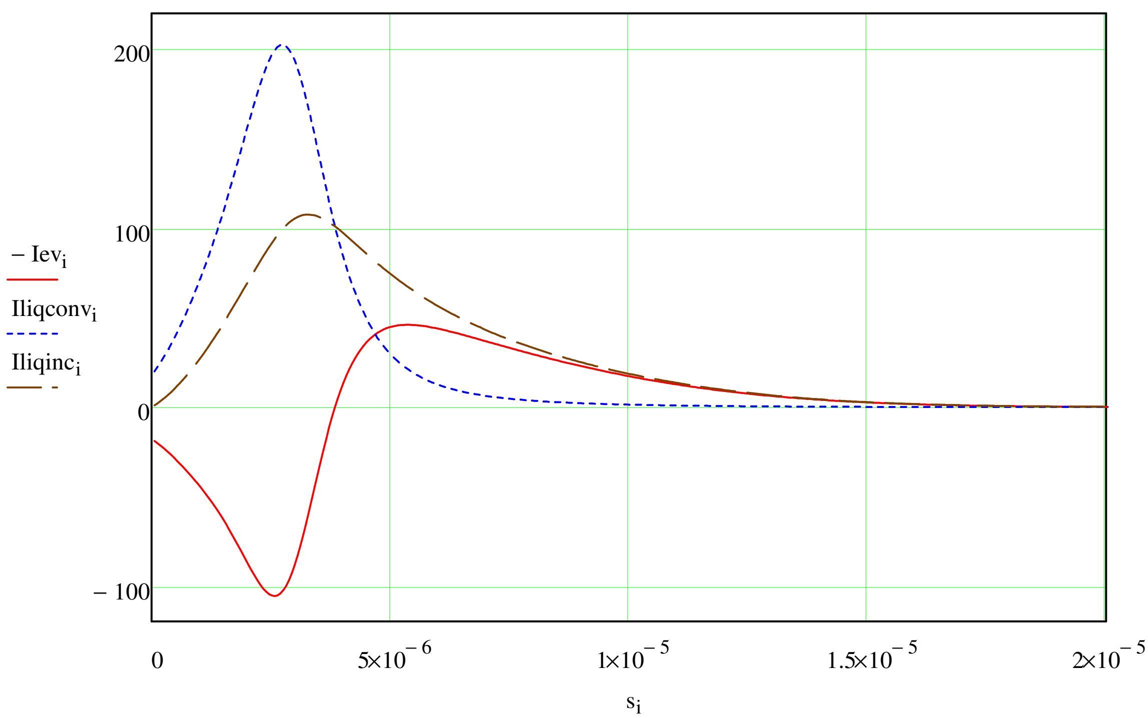
Benchmark test I, CEN. J. Claesson



Rate of evaporation



Balance for liquid water: Condensation (-Iev) + net influx (Iliqconv) = rate of increase for liquid water content (Iliqinc).



Balance for water vapor: Evaporation (Iev) + net influx (Ivapconv) =
rate of increase for water vapor content (Ivapinc).